THE FREE BOUNDARY PROBLEM FOR EULER'S EQUATIONS

by

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Abstract

We consider the motion of a perfect fluid body in vaccuum with no surface tension, in two settings. First, we study the motion of a compressible liquid occupying a bounded region of space which is subject to self-gravitational force. For this system, we construct a local-in-time solution in Sobolev spaces provided the Taylor sign condition holds initially and that the initial data satisfies certain compatibility conditions. Second, we consider the motion of an incompressible fluid subject to a uniform force of gravity which occupies an unbounded region (the water-waves system). We prove a long-time existence result for small, well-localized initial data with small initial vorticity which vanishes on the free boundary.

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Chapter 1

Introduction

The results in this thesis concern the *free boundary problem for Euler's equations*. The basic problem is the following. Suppose that at time $t = 0$, a body of fluid occupies a region $\mathcal{D}_0 \subset \mathbb{R}^3$. If the fluid is reasonably "well-behaved", what can be said about the evolution of this fluid body?

Suppose that we are considering the motion of the fluid over a time interval $[0, T]$, that the fluid occupies a region $\mathcal{D}_t$ at time t, and let $\mathcal{D} = \cup_{t\in[0,T]}\{t\} \times \mathcal{D}_t$. If the fluid is in thermodynamic equilibrium, its motion is described by three variables:

- the *velocity* vector field $v : \mathcal{D} \to \mathbb{R}^3$, which measures the velocity of the fluid particles,
- the *density* $\rho : \mathcal{D} \to \mathbb{R}_{\geq 0}$ which measures the number of particles per unit volume, and
- the *pressure* $p : \mathcal{D} \to \mathbb{R}$, whose gradient measures the forces between particles.

Neglecting the effects of viscosity, if the fluid is subject to a force $F = (F_1, F_2, F_2)$, the variables v, ρ, p evolve according to *Euler's equations*:

$$\rho\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right) v_i = -\frac{\partial}{\partial x^i} p - F_i, \qquad i = 1, 2, 3 \qquad \text{in } \mathcal{D}_t, \tag{1.0.1}$$

and *conservation of mass*:

$$\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right)\rho = -\rho \operatorname{div} v, \qquad \text{in } \mathcal{D}_t. \tag{1.0.2}$$

Here, we are writing $\operatorname{div} v = \sum_{i=1}^{3} \frac{\partial}{\partial x^i} v^i$.

If the fluid is in vaccuum and the effects of surface tension are negligible, then the pressure satisfies:

$$p = 0, \qquad \text{on } \mathcal{D}_t. \tag{1.0.3}$$

The boundary $\partial\mathcal{D}_t$ moves with the fluid velocity, so that:

$$(1, v_1, v_2, v_3) \text{ is tangent to } \partial\mathcal{D}. \tag{1.0.4}$$

One can think of (1.0.4) as a Neumann boundary condition:

$$v \cdot N = \kappa \text{ on } \partial\mathcal{D}_t, \tag{1.0.5}$$

where N is the outward-pointing unit normal vector field to $\partial\mathcal{D}_t$ and κ is the velocity of the free boundary in the direction normal to the boundary.

It was shown by Ebin [1] that the system (1.0.1)-(1.0.4) is ill-posed in Sobolev spaces without further assumptions on the initial data. We will assume that the initial data satisfies the *Taylor sign condition*:

$$\sum_{i=1}^{3} N^i \frac{\partial}{\partial x^i} p < 0 \qquad \text{on } \partial\mathcal{D}_0. \tag{1.0.6}$$

As in [2], we will see that the system (1.0.1)-(1.0.4) is locally well-posed in Sobolev spaces if the initial data satisfies (1.0.6).

The results in this thesis concern two instances of the problem (1.0.1)-(1.0.6). The first, outlined in Section 1.0.1, concerns the motion of a bounded fluid body consisting of a compressible, self-gravitating liquid, which can be thought of as a model for the motion of a star.

We prove that in this case, the problem (1.0.1)-(1.0.4) is well-posed in Sobolev spaces if the condition (1.0.6) holds. The proof can be found in Chapter 2, and is joint work with Hans Lindblad and Chenyun Luo. This is a lightly edited version of [3].

The second, outlined in Section 1.0.2, concerns a model of waves on the surface of the ocean, where the fluid domain $\mathcal{D}_t$ is assumed to be given by the graph of a function $h : \mathbb{R}^2 \to \mathbb{R}$ and the fluid is incompressible. It is now well-known that for this model, if the initial data is irrotational ($\omega \equiv \operatorname{curl} v = 0$) and sufficiently close to the equilibrium solution, then the corresponding problem (1.0.1)-(1.0.4) with $\rho \equiv 1$ and $F = g(0,0,1)$, where g is the acceleration due to gravity, admits a global-in-time solution. We consider initial data with small and well-localized vorticity ω which vanishes on the free surface boundary and relate the time of existence to the size of the initial vorticity. The proof can be found in Chapter 3 and also in [4].

1.0.1 The motion of a compressible, self-gravitating liquid with free surface boundary (joint with H. Lindblad and C. Luo)

Consider the motion of a massive fluid body. If the mass is large enough, the fluid will exert a non-trivial gravitational force on itself, and the equations (1.0.1)-(1.0.2) become:

$$\rho\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right) v_i = -\frac{\partial}{\partial x^i} p - \rho \frac{\partial}{\partial x^i}\phi, \qquad \text{in } \mathcal{D}_t, \tag{1.0.7}$$

$$\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right)\rho = -\rho \operatorname{div} v, \qquad \text{in } \mathcal{D}_t, \tag{1.0.8}$$

where here ϕ is the *gravitational potential*, defined by:

$$-\Delta\phi = C\chi_{\mathcal{D}_t}\rho, \qquad \lim_{|x|\to\infty} \phi(t,x) = 0, \tag{1.0.9}$$

where $\chi_{\mathcal{D}_t}(x) = 1$ when $x \in \mathcal{D}_t$ and $\chi_{\mathcal{D}_t}(x) = 0$ otherwise. The system (1.0.7)-(1.0.9) can be derived formally from Einstein's equations by taking the speed of light to infinity. In this section, we will consider the case that $\mathcal{D}_t$ is homeomorphic to the unit ball, so that the fluid occupies a bounded region.

The equations (1.0.7)-(1.0.8) do not form a closed system of equations, because there are four equations but five independent unknowns ρ, p, v_1, v_2, v_3. To close this sytem, we will assume that the motion is *barotropic*, meaning that the pressure is determined as a function of the mass density, $p = P(\rho)$ for some increasing function P, which for simplicity is assumed to be smooth away from $\rho = 0$. The function P is known as the *equation of state*. Since $p|_{\partial \mathcal{D}_t} = 0$ and P is invertible, it follows that $\rho|_{\partial \mathcal{D}_t} = P^{-1}(0) \equiv \overline{\rho}$ for a constant $\overline{\rho}$. If the equation of state is such that $\overline{\rho} > 0$, the fluid is said to be a *liquid*, while is $\overline{\rho} = 0$, the fluid is said to be a *gas*. The result in Chapter 2 concerns the case of a liquid. An example of such an equation of state is given by:

$$P(\rho) = c_1 \rho^\gamma - c_1 \overline{\rho}^\gamma, \tag{1.0.10}$$

for $\gamma > 0$.

The *sound speed* is defined as:

$$c_s = \sqrt{\frac{d}{d\rho} P(\rho)} \tag{1.0.11}$$

and formally taking $c_s \to \infty$ leads to an incompressible $(\operatorname{div} v = 0)$ fluid. In our result we will only consider fluids which are "almost" incompressible, in the sense that c_s is sufficiently large. Equations of state of this type were studied by Christodoulou [5] in the context of general relativity to describe the dynamics of a neutron star undergoing gravitational collapse.

It turns out that the problem (1.0.7)-(1.0.8) with boundary conditions (1.0.4)-(1.0.3) need not be solvable in Sobolev spaces for general initial data, even assuming that (1.0.6) holds. The reason is that if $\rho|_{\partial \mathcal{D}_t}$ is constant then, introducing the *material derivative* D_t:

$$D_t = \frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}, \tag{1.0.12}$$

we must have that $D_t \rho|_{\partial \mathcal{D}_t} = 0$. Restricting (1.0.8) to the boundary at $t = 0$ shows that as a consequence we must have $\operatorname{div} v|_{t=0} = 0$ on $\partial\mathcal{D}_0$, and this condition on v_0 is known as the first compatibility condition. Applying further material derivatives D_t to (1.0.7) and (1.0.8) generates higher-order conditions which need to be satisfied by the initial data in order for these equations to have a sufficiently regular solution.

To make these conditions more precise, it is convenient to work in terms of the enthalpy h, defined by:

$$h(\rho) = \int_{\bar{\rho}}^{\rho} \frac{P'(\lambda)}{\lambda}\, d\lambda, \tag{1.0.13}$$

which has the properties that $h|_{\partial\mathcal{D}_t} = 0$ and that $\partial h = \frac{1}{\rho}\partial P$ when $\partial = \frac{\partial}{\partial x}$ or $\partial = \frac{\partial}{\partial t}$. Then the equations (1.0.7)-(1.0.8) can be written as:

$$D_t v_i = -\partial_i h - \partial_i \phi, \qquad \text{in } \mathcal{D}_t, \tag{1.0.14}$$

$$D_t e(h) = \operatorname{div} v, \qquad \text{in } \mathcal{D}_t, \tag{1.0.15}$$

where here we are writing $e(h) = \log \rho(h)$, where $\rho(h)$ is determined by inverting (1.0.13).

It is also convenient to introduce *Lagrangian coordinates*. Let Ω denote the unit ball in $\mathbb{R}^3$. Then the Lagrangian coordinates are a family of maps $x(t,\cdot) : \Omega \to \mathcal{D}_t$ defined by:

$$\frac{d}{dt} x(t,y) = v(t, x(t,y)), \tag{1.0.16}$$

$$x(0,y) = x_0(y), \tag{1.0.17}$$

where here $x_0 : \Omega \to \mathcal{D}_0$ is an arbitrary diffeomorphism. In Lagrangian corodinates, the boundary $\partial\mathcal{D}_t$ is fixed:

$$\partial\mathcal{D}_t = x(t, \partial\Omega), \tag{1.0.18}$$

and the material derivative becomes the usual time derivative:

$$D_t = \frac{\partial}{\partial t}\Big|_{x=const.} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k} = \frac{\partial}{\partial t}\Big|_{y=const.} \tag{1.0.19}$$

In these coordinates, Euler's equations become:

$$D_t^2 x_i = -\partial_i h - \partial_i \phi, \qquad \text{in } [0,T]\times\Omega, \tag{1.0.20}$$

$$D_t e(h) = -\operatorname{div} v, \qquad \text{in } [0,T]\times\Omega, \tag{1.0.21}$$

where here, we are writing:

$$\partial_i = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}, \tag{1.0.22}$$

where $y(t,\cdot)$ is the inverse of the map $y \mapsto x(t,y)$.

We can now describe the compatibility conditions. Suppose that $\hat{v}(t,y) = \sum t^k v_k(x)$, and $\hat{h}(t,y) = \sum t^k h_k(y)$, are formal power series in time which satisfy the equations (1.0.7)-(1.0.8) to order M at $t = 0$:

$$D_t^k \big(D_t \hat{v}_i + \partial_i \hat{h} + \partial_i \hat{\phi}\big)\big|_{t=0} = 0, \qquad k = 0, ..., M, \tag{1.0.23}$$

$$D_t^k \big(D_t e(\hat{h}) + \operatorname{div} \hat{v}\big)\big|_{t=0} = 0, \qquad k = 0, ..., M, \tag{1.0.24}$$

where here $\hat{\phi}$ is defined by solving:

$$\Delta\hat{\phi} = -\hat{\rho}\chi_{\hat{\mathcal{D}}_t}, \text{ in } \mathbb{R}^3 \qquad \lim_{|x|\to\infty} \hat{\phi}(x) = 0, \tag{1.0.25}$$

where $\hat{\mathcal{D}}_t = \hat{x}(t,\Omega)$ and where we are writing $\hat{x}(t,y) = x_0 + \sum_{k\geq 0} t^k/k! v_k$. The equations (1.0.23)-(1.0.24) allow one to compute the coefficients $v_1, v_2, ...$ and $h_1, h_2, ...$ recursively in terms of x_0, v_0, h_0.

We say that initial data (x_0, v_0, h_0) satisfy the *compatibility conditions to order M* if:

$$h_k \in H^1_0(\Omega), \qquad k = 0, ..., M-1, \tag{1.0.26}$$

where here $H^1_0(\Omega)$ denotes the space of functions $f \in H^1(\Omega)$ which vanish on $\partial\Omega$ in the trace sense; the importance of the condition (1.0.26) is the vanishing at the boundary.

The main result in Chapter 2 is the following Theorem:

Theorem 1.0.1. *Fix $r \geq 8$ and suppose that:*

$$\mathcal{D}_0 \in H^{r+1}, \qquad v_0 \in H^r(\mathcal{D}_0), \qquad \rho_0 \in H^r(\mathcal{D}_0), \tag{1.0.27}$$

that the equation of state is sufficiently close to that of an incompressible liquid, and that the initial data satisfies the compatibility conditions of order r. Then there is a time

$$T = T(||\mathcal{D}_0||_{H^r}, ||v_0||_{H^r(\mathcal{D}_0)}, ||p_0||_{H^r(\mathcal{D}_0)}) > 0$$

so that the system (1.0.7)-(1.0.8) *has a unique solution $\mathcal{D}_t, v, \rho$ with:*

$$\mathcal{D}_t \in H^r, \qquad v(t,\cdot) \in H^{(r-1,1/2)}(\mathcal{D}_t), \qquad \rho(t,\cdot) \in H^r(\mathcal{D}_t), \tag{1.0.28}$$

and so that $(\mathcal{D}_t, v(t), \rho(t))\big|_{t=0} = (\mathcal{D}_0, v_0, \rho_0)$.

The Sobolev norm $||\cdot||_{H^{(r-1,1/2)}(\Omega)}$ is defined in Chapter 2 and measures $r-1$ full derivatives and $1/2$ a tangential derivative in L^2. Here, we are writing $\mathcal{D}_t \in H^r$ to mean that locally, $\mathcal{D}_t$ can be written as the graph of a function in $H^r(\mathbb{R}^3)$.

The method we use builds on [6] and [7]; and involves constructing solutions to a smoothed-out version of the equations and uniform energy estimates. One difficulty present in this problem that is not present in the case of an incompressible fluid or a compressible gas is that the smoothed-out problem has its own set of compatibility conditions which need to be satisfies in order to prove existence and it is hard to construct this data. For a more detailed discussion, see Chapter 2.

1.0.2 Gravity water waves with vorticity

We now consider the following idealized model for the motion of waves on the surface of the ocean. Consider the case that $\mathcal{D}_t$ is diffeomorphic to the lower-half space $\mathbb{R}^3_- = \{(x_1, x_2, y) | y \leq 0\}$. If the fluid is incompressible and subject to a uniform force of acceleration in the vertical direction, the equations (1.0.1)-(1.0.2) become:

$$\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right) v_i = -\partial_i p - g e_3, \qquad \text{in } \mathcal{D}_t, \tag{1.0.29}$$

$$\operatorname{div} v = 0, \qquad \text{in } \mathcal{D}_t. \tag{1.0.30}$$

Here, g is the acceleration due to gravity and $e_3 = (0, 0, 1)$. We will assume here that $\mathcal{D}_t$ is given by the graph of a function h, $\mathcal{D}_t = \{(x_1, x_2, y) | y \leq h(t, x_1, h_2)\}$, which is sufficient to describe sufficiently small perturbations of still water.

In this case, the unit normal N to the free boundary $\mathcal{D}_t$ is given by:

$$N = \frac{1}{\sqrt{1 + |\nabla h|^2}} (\partial_{x_1} h, \partial_{x_2} h, -1), \qquad \nabla = (\partial_{x_1}, \partial_{x_2}), \tag{1.0.31}$$

and the boundary condition (1.0.4) becomes the following evolution equation for h:

$$\partial_t h = \partial_{x_1} h v^1 + \partial_{x_2} h v^2 - v^3. \tag{1.0.32}$$

The majority of work on the equations (1.0.29)-(1.0.30) with free boundary has concerned the irrotational case, $\operatorname{curl} v = 0$. In this case, $v = \nabla_{x,y} \psi$ for a harmonic function $\Delta \psi = 0$. The fluid motion is then entirely determined by h and $\varphi \equiv \psi|_{\partial \mathcal{D}_t}$. A calculation using the chain rule (which can be found in [8] as well as in Section 3.3 of this thesis) shows that h, φ satisfy the following system:

$$\partial_t \varphi = -gh + |\nabla \varphi|^2 + \frac{1}{2(1+|\nabla h|^2)}(G(h)\varphi + \nabla h \cdot \nabla \varphi)^2, \tag{1.0.33}$$

$$\partial_t h = G(h)\varphi. \tag{1.0.34}$$

Here, we are writing $\nabla = (\partial_{x^1}, \partial_{x^2})$ and writing $G(h)$ for the rescaled Dirichlet-to-Neumann map, defined by:

$$G(h)f = \left(1+|\nabla h|^2\right)^{-1/2} \partial_N F|_{\partial \mathcal{D}_t}, \qquad \text{where } \Delta F = 0 \text{ in } \mathcal{D}_t, F|_{\partial \mathcal{D}_t} = f, \tag{1.0.35}$$

with $\partial_N = N \cdot \partial$. Let $\Lambda = |\nabla|$ be the operator given by:

$$\mathcal{F}(\Lambda f)(\xi) = |\xi| \mathcal{F} f(\xi), \qquad \mathcal{F} f(\xi) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} f(x)\, dx. \tag{1.0.36}$$

Inserting the following formal expansion of $G(h)$ in powers of h (see, for example, [8]):

$$G(h)f = \Lambda f - \nabla \cdot (h \nabla f) - \Lambda(h \Lambda f) + O(h^2) \tag{1.0.37}$$

into the system (1.0.33)-(1.0.34) leads to:

$$\partial_t \varphi = -h - \frac{1}{2}|\nabla \varphi|^2 + \frac{1}{2}|\Lambda \varphi|^2 + ... \tag{1.0.38}$$

$$\partial_t h = \Lambda \varphi - \nabla \cdot (h \nabla \varphi) - \Lambda(h \Lambda \varphi)... \tag{1.0.39}$$

In terms of the complex variable $u \equiv h + i\Lambda^{1/2}\varphi$, the system (1.0.38)-(1.0.39) can then be written as:

$$(\partial_t + i\Lambda^{1/2})u = N(u, \overline{u}), \qquad \overline{u} = h - i\Lambda^{1/2}\varphi, \tag{1.0.40}$$

where N is a nonlinear, nonlocal function of u which depends on derivatives of its arguments.

The equation (1.0.40) is a quasilinear dispersive equation for which solutions to the linearized equation:

$$(\partial_t + i\Lambda^{1/2})u_{lin} = 0 \tag{1.0.41}$$

decay at the rate of $t^{-d/2}$, where $d = 1, 2$ is the dimension of the fluid interface. A discussion of the history this problem can be found in Chapter 3, but summarizing briefly, the system (1.0.33) -(1.0.34) has global-in-time solutions for sufficiently small and well-localized initial data for both one- and two-dimensional fluid interfaces and with additional physcial effects (see [9], [10], [11], as well as [12] for a survey article).

On the other hand, little is known about the long-time behavior of the system (1.0.29)-(1.0.30) in the case that $\operatorname{curl} v \neq 0$. In the case without free boundary:

$$\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right) v_i = -\frac{\partial}{\partial x^i} p, \qquad \text{in } \mathbb{R}^3, \tag{1.0.42}$$

$$\operatorname{div} v = 0, \qquad \text{in } \mathbb{R}^3, \tag{1.0.43}$$

the groundbreaking result of Beale-Kato-Majda [13] showed that the vorticity $\omega \equiv \operatorname{curl} v$ is the only obstruction to obtaining regular solutions to the incompressible Euler equations (1.0.42)-(1.0.43) given regular initial data. Specifically, they prove that for $s > 3/2 + 1$, if $v \in C([0, T); H^s(\mathbb{R}^3))$ satisfies (1.0.42)-(1.0.43) and breaks down as $t \to T$, then

$$\lim_{t \nearrow T} \int_0^T ||\omega(s)||_{L^\infty(\mathbb{R}^3)}\, ds = \infty. \tag{1.0.44}$$

In particular, this shows that irrotational solutions to (1.0.42)-(1.0.43) have global lifespans.

Part of the difficulty in dealing with the long-time behavior of fluids with nonzero vorticity is that the vorticity satisfies the following nonlinear transport equation:

$$\left(\frac{\partial}{\partial t} + \sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k}\right) \omega = (\partial v) \cdot \omega. \tag{1.0.45}$$

Thinking of $\partial v \sim \omega$ and ignoring the transport term $\sum_{k=1}^{3} v^k \frac{\partial}{\partial x^k} \omega$ on the left-hand side suggests that solutions to the equation (1.0.45) should behave like solutions to the ODE:

$$\frac{d}{dt}Y = Y^2, \tag{1.0.46}$$

and this blows up as t approaches $\frac{1}{Y(0)}$.

In a similar vein, Ionescu and Lie [14], considered the Euler-Maxwell "one-fluid" model and prove that if the initial vorticity ω_0 satisfies $||\omega_0|| \leq \delta$ in a certain (weighted Sobolev) norm $||\cdot||$ for small δ, then small, well-localized initial data leads to a solution which exists until at least:

$$T \sim \frac{1}{\delta}. \tag{1.0.47}$$

In particular this proves that if $\omega_0 = 0$, the solution persists for all time. Note also that the lifespan (1.0.47) is consistent with the lifespan of the ODE (1.0.46).

The result in Chapter 3 concerns the long-time behavior of the system (1.0.29)-(1.0.30) with boundary conditions (1.0.3),(1.0.4) with nonzero vorticity. We start by decomposing the vector field v into its irrotational and rotational part:

$$v = \nabla_{x,y}\psi + v_\omega, \tag{1.0.48}$$

where:

$$\Delta\psi = 0, \qquad \text{in } \mathcal{D}_t, \tag{1.0.49}$$

$$N \cdot \nabla_{x,y}\psi = N \cdot v, \qquad \text{on } \partial\mathcal{D}_t, \tag{1.0.50}$$

and

$$\operatorname{curl} v_\omega = \omega, \qquad \text{in } \mathcal{D}_t, \tag{1.0.51}$$

$$\operatorname{div} v_\omega = 0, \qquad \text{in } \mathcal{D}_t, \tag{1.0.52}$$

$$N \cdot v_\omega = 0, \qquad \text{on } \partial\mathcal{D}_t. \tag{1.0.53}$$

We also define $\varphi = \psi|_{\partial\mathcal{D}_t}$ and write $u = \varphi + i\Lambda^{1/2}h$. Then the main result of Chapter 3 is:

Theorem 1.0.2. *Fix $N_1 \geq 6, N >> 1$ and define $N_0 = NN_1$. There are $\varepsilon_0^*, \varepsilon_1^* > 0$ with $\varepsilon_1^* << \varepsilon_0^*$ with the following property. If (v_0, h_0) are initial data satisfying the bound:*

$$||v_0||_{H^{N_0}(\mathcal{D}_0)} + ||h_0||_{H^{N_0}(\mathbb{R}^2)} + ||xu_0||_{L^2(\mathbb{R}^2)} \leq \frac{1}{2}\varepsilon_0, \quad \varepsilon_0 < \varepsilon_0^*, \tag{1.0.54}$$

and $\omega_0 = \operatorname{curl} v_0$ satisfies:

$$\omega_0|_{\partial\mathcal{D}_0} = 0, \tag{1.0.55}$$

and:

$$\sum_{j \leq N_1} \int_{\mathcal{D}_0} (1 + |x|^2 + y^2)^6 |\nabla_{x,y}^j \omega_0(x,y)|^2 \, dxy \leq \frac{1}{2}\varepsilon_1, \quad \varepsilon_1 < \varepsilon_1^*, \tag{1.0.56}$$

then the system (1.0.29)-(1.0.30) with initial data $v|_{t=0} = v_0, h|_{t=0} = h_0$ and boundary conditions (1.0.4)-(1.0.3) has a solution

$$v(t,\cdot) \in H^{N_0}(\mathcal{D}_t), \qquad h(t,\cdot) \in H^{N_0}(\mathbb{R}^2), \tag{1.0.57}$$

for $t \in [0, T_*]$ where:

$$T_* \geq C_N \min\left(\frac{1}{\varepsilon_1^{1/2}}, \frac{1}{\varepsilon_0^N}\right). \tag{1.0.58}$$

Note that in particular, if one takes $\varepsilon_1 = 0$, this gives a lower bound on the lifespan of solutions of:

$$T_* \geq C_N \frac{1}{\varepsilon_0^N}, \tag{1.0.59}$$

where N may be taken arbitrarily large if the initial data is taken arbitrarily smooth.

This appears to be the first result concerning the long-time behavior of solutions to (1.0.29)-(1.0.30) in three space dimensions with nonzero vorticity. Comparing this result to other results in the literature (especially [14] and [10]), one should expect a lifespan of:

$$T_* \sim \frac{1}{\varepsilon_1}. \tag{1.0.60}$$

There are two main difficulties with this which are not present in [14]. The first is that in [14], solutions to the corresponding linearized problem decay at a rate $t^{-1+\beta}$ for small β instead to the "almost integrable" rate t^{-1}. The second difficulty is that the vorticity enters into the right-hand side of the equation for the dispersive variable u *linearly* which leads to a shorter lifespan. I plan to address both of this issues in the future.

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Chapter 2

Local Well-Posedness for the Motion of a Compressible, Self-Gravitating Liquid with Free Surface Boundary (joint with H. Lindblad and C. Luo)

2.1 Introduction

The motion of a barotropic, self-gravitating fluid occupying a region $\mathcal{D} = \cup_{0\leq t\leq T}\{t\} \times \mathcal{D}_t, \mathcal{D}_t \subset \mathbb{R}^3$, of space time, is described by the velocity $V = (V^1, V^2, V^3)$, a non-negative function ρ known as the density, and an equation of state $p = p(\rho)$ which specifies the pressure p as a function of ρ, and which is assumed to be non-negative and strictly increasing. The equations of motion are then given by Euler's equations:

$$\rho(\partial_t + V^k\partial_k)v_i + \partial_i p + \rho\partial_i\phi = 0, \text{ for } i = 1, 2, 3 \text{ in } \mathcal{D}, \tag{2.1.1}$$

and the continuity equation:

$$(\partial_t + V^k\partial_k)\rho + \rho \operatorname{div} V = 0 \text{ in } \mathcal{D}, \tag{2.1.2}$$

where repeated upper and lower indices are summed over, $\partial_i = \partial/\partial x^i$, $v_i = V^i$ and $\text{div} V = \partial_i V^i$. Here, with $\chi_{\mathcal{D}_t}$ the characteristic function of $\mathcal{D}_t$, the Newtonian gravity potential ϕ is defined to be the unique solution to:

$$\Delta\phi = -\rho\chi_{\mathcal{D}_t}, \qquad \text{in } \mathbb{R}^3, \quad \text{with} \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{2.1.3}$$

Explicitly, ϕ is given by:

$$\phi(t,x) = \frac{1}{4\pi} \int_{\mathcal{D}_t} \frac{\rho(t,x')\, dx'}{|x - x'|}. \tag{2.1.4}$$

Particles on the boundary $\partial\mathcal{D}_t$ move with the velocity of the fluid, and if the body moves in vacuum then the pressure vanishes outside of $\mathcal{D}$, so we also require the boundary conditions:

$$(\partial_t + V^k\partial_k)\big|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \tag{2.1.5}$$

$$p = 0, \qquad \text{on } \partial\mathcal{D}_t, \tag{2.1.6}$$

where $\partial\mathcal{D} = \cup_{0\le t\le T}\partial\mathcal{D}_t$ is the space time boundary. Since the equation of state $p(\rho)$ is strictly increasing, we can alternatively think of the density as a function of the pressure and then (2.1.6) implies that $\rho|_{\partial\mathcal{D}_t} = \overline{\rho}$ for some constant $\overline{\rho}$, with $p(\overline{\rho}) = 0$. We consider the case $\overline{\rho} > 0$, in which case the fluid is said to be a liquid.

Given an open set $\mathcal{D}_0 \subset \mathbb{R}^3$ and a diffeomorphism $x_0 : \Omega \to \mathcal{D}_0$ from the unit ball Ω, a function ρ_0 which is strictly positive on $\mathcal{D}_0$ so that $p(\rho_0) = 0$ on $\partial\mathcal{D}_0$, and a vector field V_0 on $\mathcal{D}_0$, the *free boundary problem for the compressible Euler equations in a bounded domain* is to find a domain $\mathcal{D} = \cup_{0\le t\le T}\{t\} \times \mathcal{D}_t$, a vector field V and a function ρ satisfying (2.1.1)-(2.1.6) as well as the initial conditions:

$$\{x : (0,x) \in \mathcal{D}\} = \mathcal{D}_0, \quad \text{and} \quad V(0,x) = V_0(x), \quad \rho(0,x) = \rho_0(x), \quad \text{in } \mathcal{D}_0. \tag{2.1.7}$$

Since ρ is constant on the boundary it follows that $(\partial_t + V^k\partial_k)\rho = 0$ on the boundary so by (2.1.2) at $t=0$ we must have that $\operatorname{div} V_0 = 0$ on the boundary. Similarly $(\partial_t + V^k\partial_k)^2\rho = 0$ on the boundary, which by (2.1.2) implies that $(\partial_t + V^k\partial_k)\operatorname{div} V = 0$ on the boundary, but taking the divergence of (2.1.1) gives an expression for $(\partial_t + V^k\partial_k)\operatorname{div} V$ in terms of space derivatives of V and ρ, and this expression must vanish on the boundary. We say that the initial data V_0, ρ_0 satisfy the *compatibility condition of order m* if there are formal power series in t, $\hat{\rho}(t,x), \hat{V}(t,x), \hat{\phi}(t,x)$ that satisfy (2.1.1)-(2.1.3) with $\hat{V}(0,x) = V_0(x)$, $\hat{\rho}(0,x) = \rho_0(x)$ and:

$$(\partial_t + \hat{V}^k\partial_k)^j(\hat{\rho} - \overline{\rho}) \in H_0^1(\mathcal{D}_0), \quad \text{for } j = 0, \ldots, m. \tag{2.1.8}$$

In addition, this problem is ill-posed (see [1]) unless the physical (Taylor) sign condition holds:

$$-\nabla_N p \geq \delta > 0, \text{ on } \partial\mathcal{D}_t, \qquad \text{where } \nabla_N = N^i\nabla_i. \tag{2.1.9}$$

Our apriori bounds hold in general but for the existence of a solution satisfying the compatibility conditions we need to assume that we are close to the incompressible case, i.e. $\rho(p)$ is close to the constant function:

$$0 < \rho'(p) \leq \delta_0, \quad \text{and} \quad |\rho^{(k)}(p)| \leq \delta_0/E_0^{k-1}, \quad k = 2, \ldots, r, \tag{2.1.10}$$

where:

$$E_0 = ||V_0||_{H^r} + ||\rho_0||_{H^r}. \tag{2.1.11}$$

Our main result is:

Theorem 2.1.1. *There is a constant $\delta_0 > 0$ such that if $\rho(p)$ is a smooth function satisfying $\rho(0) > 0$ and (2.1.10) for $k = 1$ the following hold. Suppose that there is a diffeomorphism $x_0 : \Omega \to \mathcal{D}_0$ in $H^{r+1}(\Omega)$ and that $V_0, \rho_0 \in H^r(\mathcal{D}_0)$ satisfy the compatibility conditions to order $r - 1 \geq 7$, and that (2.1.9) holds at $t = 0$ and that (2.1.10) hold for $k = 2,...,r$. Then there is*

$T = T(||x_0||_{H^r}, ||V_0||_{H^r}, ||\rho_0||_{H^r}) > 0$ *so that* (2.1.1)-(2.1.6) *has a solution* $(V, \rho, \mathcal{D})$ *with diffeomorphisms* $x(t,\cdot): \Omega \to \mathcal{D}_t$ *in* $H^r(\Omega)$ *and* $V(t,\cdot) \in H^{(r-1,1/2)}(\mathcal{D}_t)$, $\rho(t,\cdot) \in H^r(\mathcal{D}_t)$, *for* $0 \le t \le T$.

The space $H^{(r-1,1/2)}$ defined in (2.3.11) controls $r-1$ full derivatives and half a tangential derivative. While the density ρ is as regular at later times as at $t=0$, we need to assume more regularity of V and the diffeomorphisms $x(t,\cdot)$ initially than we get back at later times. We will however prove energy estimates controlling $x(t,\cdot) \in H^r(\Omega)$ and $V(t,\cdot) \in H^{(r-1,1/2)}(\mathcal{D}_t)$, $\rho(t,\cdot) \in H^r(\mathcal{D}_t)$ for $t \le T$ in terms of these quatities at $t=0$. In the incompressible case, when the compatibility conditions are automatically satisfied, one can regularize the initial data and use the energy estimates to prove existence in the energy space, see [2].

Related problems without self gravity have previously been solved using different methods. In [3], Wu proved local well-posedness for the incompressible ($\operatorname{div} v = 0$) irrotational ($\operatorname{curl} v = 0$) case, using complex analysis and spinors. Lindblad [4, 5] used a Nash-Moser iteration scheme to solve the case with $\operatorname{curl} v \neq 0$ without self-gravity in the incompressible case and the case of a compressible liquid. Later, Coutand-Shkoller [6] and Coutand-Hole-Shkoller [7], were able to use tangential smoothing together with surface tension and artificial viscosity, assuming the elliptic estimates proven in [8] to avoid the use of a Nash-Moser iteration.

Lindblad-Nordgren [9] proved apriori bounds for an incompressible liquid with self gravity in the two dimensional case. Nordgren [2] proved local existence for an incompressible liquid with self gravity in the three dimensional case. His proof built on the approach of [6] but he was able to avoid the need for artificial viscosity and surface tension using elliptic estimates from [9]. Here we prove local well-posedness for a compressible liquid with self gravity. The method here builds on [9, 2, 4, 5, 6, 7]. In particular we use tangential smoothing but we avoid any extra smoothing by surface tension or artificial viscosity, by using improved elliptic estimates and estimates for a wave equation on a bounded domain that we prove.

2.1.1 The setup for the proof

We fix Ω to be the unit ball in $\mathbb{R}^3$ and a diffeomorphism $x_0 : \Omega \to \mathcal{D}_0$. We introduce Lagrangian coordinates, see Section 2.2, so the boundary is fixed:

$$\frac{dx}{dt} = V(t,x), \qquad x(0,y) = x_0(y), \qquad y \in \Omega. \tag{2.1.12}$$

We express Euler's equations in these coordinates, using the enthalpy, $h'(\rho) = p'(\rho)/\rho$, $h(\bar{\rho}) = 0$,

$$D_t v_i = -\partial_i h - \partial_i \phi, \quad \text{in} \quad [0,T] \times \Omega, \quad \text{where} \quad D_t = \partial_t\big|_{y=const}, \quad \partial_i = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}. \tag{2.1.13}$$

If we take the material derivative D_t of the continuity equation $D_t\rho = -\rho \operatorname{div} V$ and the divergence of Euler's equations (2.1.13), using (2.1.3), we obtain, with $e(h) = \log \rho(h)$,

$$D_t^2 e(h) - \Delta h = (\partial_i V^j)(\partial_j V^i) - \rho(h), \quad \text{in } [0,T]\times\Omega, \quad \text{with} \quad h\big|_{[0,T]\times\partial\Omega} = 0, \tag{2.1.14}$$

where $\Delta = \delta^{ij}\partial_i\partial_j$. Here ϕ is given by

$$\phi(t,y) = \frac{1}{4\pi}\int_\Omega \frac{\rho(t,y')\kappa(t,y')\,dy'}{|x(t,y) - x(t,y')|}, \qquad \text{where} \qquad \kappa = |\det(\partial x/\partial y)|. \tag{2.1.15}$$

It is possible to obtain apriori energy bounds for the system (2.1.13)-(2.1.14) but it is difficult to come up with an iteration scheme that doesn't lose regularity. We will first smooth out the equations. Let $S_\varepsilon = T_\varepsilon^* T_\varepsilon$ be a regularization in directions tangential to the boundary that is self adjoint, see Section 2.3.1. Given a velocity vector field V, we define the tangentially regularized velocity and the regularized coordinates by

$$\widetilde{V} = S_\varepsilon V, \qquad \frac{d\widetilde{x}}{dt} = \widetilde{V}(t,y), \qquad \widetilde{x}(0,y) = x_0(y), \qquad y \in \Omega. \tag{2.1.16}$$

Using these regularized coordinates we defined the smoothed out equations by

$$D_t v_i = -\widetilde{\partial}_i h - \widetilde{\partial}_i \phi, \quad \text{in} \quad [0,T]\times\Omega, \quad \text{where} \quad D_t = \partial_t\big|_{y=const}, \quad \widetilde{\partial}_i = \frac{\partial y^a}{\partial \widetilde{x}^i}\frac{\partial}{\partial y^a}, \tag{2.1.17}$$

where h is given by

$$D_t^2 e(h) - \widetilde{\Delta} h = (\widetilde{\partial}_i \widetilde{V}^j)(\widetilde{\partial}_j V^i) - \rho(h) \quad \text{in } [0,T]\times\Omega, \quad \text{with} \quad h\big|_{[0,T]\times\partial\Omega} = 0, \tag{2.1.18}$$

where:

$$\widetilde{\Delta} = \delta^{ij}\widetilde{\partial}_i\widetilde{\partial}_j, \tag{2.1.19}$$

and ϕ is given by

$$\phi(t,y) = \frac{1}{4\pi}\int_\Omega \frac{\rho(t,y')\widetilde{\kappa}(t,y')\,dy'}{|\widetilde{x}(t,y)-\widetilde{x}(t,y')|}, \qquad \text{where} \qquad \widetilde{\kappa} = |\det(\partial\widetilde{x}/\partial y)|. \tag{2.1.20}$$

Taking the divergence of (2.1.17) and subtracting it from (2.1.18) shows that $D_t\rho = -\rho \operatorname{div} v$ if this holds at $t=0$.

One can prove uniform apriori energy bounds for the system (2.1.16)-(2.1.18) up to a time $T>0$, independent of ε. Moreover, one can prove ε dependent bounds for the iteration scheme: given V, define $\tilde{V}$ and $\widetilde{x}$ by (2.1.16), and then h and the new V by solving the system (2.1.17)-(2.1.18). We will show in Theorem 2.9.1 that this system has a unique solution on a time interval of size ε. In Theorem 2.12.1, we show the solutions satisfy energy estimates which are uniform in ε. This allows us to extend the solution to a time independent of ε, and by taking the limit as $\varepsilon\to 0$ obtain a solution to the original system (2.1.13)-(2.1.14); see Section 2.4.

2.1.2 Energy estimates

Let E be the energy for Euler's equations:

$$E(t) = \int_{\mathcal{D}_t}(|v|^2 + Q(\rho) + \phi)\,\rho\,dx = \int_\Omega (|v|^2 + Q(\rho) + \phi)\,\rho\kappa dy, \tag{2.1.21}$$

where:

$$Q(\rho) = 2\int p(\rho)\rho^{-2} d\rho. \tag{2.1.22}$$

If we take the time derivative of the integral expressed in the fixed Lagrangian coordinates we get D_t applied to the integrand. We then use Euler's equation $D_t v_i = -\rho^{-1}\partial_i p - \partial_i \phi$ and integrate by parts:

$$\begin{aligned}\frac{dE}{dt} &= \int_{\mathcal{D}_t} 2V^i(-\partial_i p - \rho\partial_i\phi) - Q'(\rho)\rho D_t\rho + \rho D_t\phi\, dx \\ &= \int_{\mathcal{D}_t} 2\,\mathrm{div}V p - Q'(\rho)\rho D_t\rho + (D_t\phi - 2V^i\partial_i\phi)\rho\, dx + \int_{\partial\mathcal{D}_t} 2v_i p N^i dS. \end{aligned} \tag{2.1.23}$$

Using the continuity equation $D_t\rho = -\rho\,\mathrm{div}\,V$ and the boundary condition $p=0$ only terms with ϕ remain. Let us for simplicity assume that $\rho\kappa = 1$, and let $\Phi(z) = (4\pi|z|)^{-1}$. With $\Phi_i(z) = \partial_i\Phi(z) = -\Phi_i(-z)$ we have

$$\begin{aligned}\int_{\mathcal{D}_t} D_t\phi\,\rho\, dx &= \iint_{\Omega\times\Omega} D_t\Phi\big(x(t,y) - x(t,y')\big)\,dy dy' \\ &= \iint_{\Omega\times\Omega} \big(V(t,y) - V(t,y')\big)\Phi_i\big(x(t,y) - x(t,y')\big)\,dy dy' = \int_{\mathcal{D}_t} 2V^i\partial_i\phi\,\rho\, dx. \end{aligned} \tag{2.1.24}$$

It follows that $E'(t) = 0$. This energy for the smoothed problem with $\mathcal{D}_t, dx$ replaced by $\widetilde{\mathcal{D}}_t, d\widetilde{x}$ is almost conserved apart from that $D_t(\rho\widetilde{\kappa}) = \rho\widetilde{\kappa}(\mathrm{div}\widetilde{V} - \mathrm{div}V)$. We will obtain energies for derivatives of the smoothed problem which will contain a boundary term where the symmetry of the smoothing matters, see Section 2.10.

2.2 Lagrangian Coordinates and the wave equation for the enthalphy

Let Ω be the unit ball in $\mathbb{R}^3$ and $x_0 : \Omega \to \mathcal{D}_0$ be a diffeomorphism. Suppose that $v(t,x), p(t,x), \rho(t,x)$ satisfy (2.1.1)-(2.1.6). The Lagrangian coordinates $x(t,y)$ are given by:

$$\frac{d}{dt}x(t,y) = V(t,x(t,y)), \qquad x(0,y) = x_0(y), \qquad y \in \Omega. \tag{2.2.1}$$

We define the material derivative:

$$D_t f(t,y) = \frac{\partial}{\partial t}\Big|_{y=constant} f(t,y). \tag{2.2.2}$$

We will use the letters $i, j, k \dots$ to refer to quantities expressed in terms of the usual Eulerian coordinates and $a, b, c, \dots$ to refer to quantities expressed in Lagrangian coordinates, e.g.

$$\partial_i = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a} = \frac{\partial y^a}{\partial x^i}\partial_a. \tag{2.2.3}$$

In these coordinates we can now write Euler's equations (2.1.1) and the continuity equation (2.1.2) as

$$\rho D_t V^i = -\delta^{ij}(\partial_i p + \rho \partial_i \phi), \quad \text{on } [0,T] \times \Omega, \tag{2.2.4}$$

$$D_t \rho = -\rho \operatorname{div} V, \qquad\qquad \text{on } [0,T] \times \Omega, \tag{2.2.5}$$

where $\operatorname{div} V = \partial_i V^i$ and ∂_i acts on functions defined on Ω by (2.2.3), where x is obtained from V by (2.2.1). Writing $\kappa = \det(\partial x / \partial y)$, by (2.2.5) and the formula for the derivative of the determinant, we have $D_t \kappa = \kappa \operatorname{div} V$.

The gravitational potential is then given in terms of the fundamental solution of the

Laplacian by:

$$\phi(t,y) = \frac{1}{4\pi}\int_{\mathcal{D}_t} \frac{\rho(t,y(t,x'))\,dx'}{|x(t,y)-x'|} = \frac{1}{4\pi}\int_{\Omega} \frac{\rho(t,y')\kappa(t,y')\,dy'}{|x(t,y)-x(t,y')|}$$

$$= \frac{1}{4\pi}\int_{\Omega} \frac{\rho_0(y')\kappa_0(y')\,dy'}{|x(t,y)-x(t,y')|}. \tag{2.2.6}$$

2.2.1 The enthalpy formulation

The pressure is determined from the mass density, $p = p(\rho)$ for a smooth, increasing function p and we can alternatively think of $\rho = \rho(p)$. With $\bar{\rho} = \rho^{-1}(0)$, we define the enthalpy by:

$$h(\rho) = \int_{\bar{\rho}}^{\rho} \frac{p'(\lambda)}{\lambda} d\lambda. \tag{2.2.7}$$

We then have $\partial_i p = \rho\, \partial_i h$, so (2.2.4) becomes:

$$D_t V^i = -\delta^{ij}(\partial_j h + \partial_j \phi). \tag{2.2.8}$$

Since we assume that $p'(\lambda) > 0$, the function $\rho \to h(\rho)$ is invertible. We can then write $\rho = \rho(h)$ and think of h as the fundamental thermodynamic quantity. Defining $e(h) = \log \rho(h)$, we can re-write (2.2.5) in terms of h:

$$D_t e(h) + \operatorname{div} V = 0. \tag{2.2.9}$$

Taking the divergence of (2.2.8) and the time derivative of (2.2.9) using that $[D_t, \partial_j] = -(\partial_j V^k)\partial_k$, we get:

$$D_t^2 e(h) - \Delta h = (\partial_i V^j)(\partial_j V^i) - \rho(h), \text{ in } [0,T]\times\Omega, \quad \text{with} \quad h = 0, \text{ on } [0,T]\times\partial\Omega, \tag{2.2.10}$$

Here, ∂_i is given by (2.2.3), Δ is the Laplacian on Ω induced by the coordinates x and the flat metric on $\mathbb{R}^3$:

$$\Delta h = \delta^{ij}\partial_i\partial_j h = \kappa^{-1}\partial_a\big(\kappa g^{ab}\partial_b h\big), \quad \text{where} \quad g^{ab} = \delta^{ij}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}, \quad \text{and} \quad \kappa = \det(\partial x/\partial y). \tag{2.2.11}$$

On the other hand, starting with (2.2.10) and taking the divergence of (2.2.8), (2.2.9) is automatically satisfied if it is satisfied at $t = 0$. By (2.2.7) respectively (2.2.9), we have that:

$$h\big|_{t=0} = e^{-1}(\log\rho_0) \equiv h_0, \quad \text{and} \quad D_t h\big|_{t=0} = -\operatorname{div} V_0/e'(h_0) \equiv h_1, \qquad \text{in } \Omega. \tag{2.2.12}$$

Assuming that the initial-boundary value problem (2.2.10)-(2.2.12) has a unique solution h for given V, the initial-free boundary problem for Euler's equations (2.1.1)-(2.1.7) is equivalent to the fixed boundary problem:

$$D_t V^i = -\delta^{ij}\partial_j h - \delta^{ij}\partial_j\phi, \quad \text{in } [0,T]\times\Omega, \tag{2.2.13}$$

$$D_t x^i = V^i, \quad \text{in } [0,T]\times\Omega, \tag{2.2.14}$$

$$D_t^2 e(h) - \Delta h = (\partial_i V^j)(\partial_j V^i) - \rho(h), \quad \text{in } [0,T]\times\Omega, \tag{2.2.15}$$

$$h = 0 \quad \text{on } [0,T]\times\partial\Omega, \tag{2.2.16}$$

$$x(0,y) = x_0(y), D_t x(0,y) = V_0(y), h(0,y) = h_0, D_t h(0,y) = h_1. \tag{2.2.17}$$

2.2.1.1 Assumptions on the equation of state

With $\delta_0 > 0$ as in Appendix 2.E let $c_1 > 0$ be a constant such that

$$0 < c_1 \le e'(h) \le \delta_0, \quad \text{and} \quad |e^{(k)}(h)| \le \delta_0/E_0^{k-1}, \quad k = 2,\dots,r, \tag{2.2.18}$$

where:

$$E_0 = ||V_0||_{H^r} + ||\rho_0||_{H^r}. \tag{2.2.19}$$

2.2.2 Higher order commutators

Repeatedly using that $[D_t, \partial_j] = -(\partial_j V^\ell)\partial_\ell$ it follows that

$$D_t^k \partial_i = \sum_{\ell \le k} S_{i\ell}^{jk} \partial_j D_t^\ell, \tag{2.2.20}$$

where $S_{ik}^{jk} = \delta_i^j$, and for $\ell < k$, we have for some constants $c_{\ell\ell_1\dots\ell_n}^{kn}$

$$S_{i\ell}^{jk} = S_{i\ell}^{jk}(\partial V, \dots, \partial D_t^{k-\ell-1} V) = c_{\ell\ell_1\dots\ell_n}^{kn} (\partial_i D_t^{\ell_1} V^{i_2}) \cdots (\partial_{i_n} D_t^{\ell_n} V^j), \tag{2.2.21}$$

where the sum is over $\ell_1 + \cdots + \ell_n = k - \ell - n$ and $n = 1, \dots k$. Here the terms with $n = 1$ should be interpreted as $c_{\ell\ell'}^{k1}(\partial_i D_t^{\ell'} V^j)$ and the terms with $n = 2$ should be interpreted as $c_{\ell\ell'\ell''}^{k2}(\partial_i D_t^{\ell'} V^{i''})(\partial_{i''} D_t^{\ell''} V^j)$.

The potential $\phi = \Phi[\rho\kappa]$ can be expressed in terms an integral operator

$$\Phi[f](t,y) = \int_\Omega K(t,y,y') f(t,y')\, dy', \quad \text{where} \quad K(t,y,y') = \frac{1}{4\pi} \frac{1}{|x(t,y) - x(t,y')|}. \tag{2.2.22}$$

$D_t^k \Phi[f]$ is a sum of integral operators $\Phi_\ell[D_t^{k-\ell} f]$, $\ell \le k$, with kernels that are sums over $\ell_1 + \cdots + \ell_n = \ell$ of

$$K_\ell(\delta x, \delta V, \dots, \delta D_t^{\ell-1} V) = d_{\ell_1 \dots \ell_n}^{\ell} \frac{(\delta D_t^{\ell_1} x \cdot \delta D_t^{\ell_2} x) \cdots (\delta D_t^{\ell_{n-1}} x \cdot \delta D_t^{\ell_n} x)}{|x(t,y) - x(t,y')|}, \tag{2.2.23}$$

where

$$\delta W(t,y,y') = \frac{W(t,y) - W(t,y')}{|x(t,y) - x(t,y')|}. \tag{2.2.24}$$

2.2.3 The compatibility conditions

The compatibility condition of order m (2.1.8) can now be expressed in the Lagrangian coordinates as that the formal power series solution in t: $\hat{V}(t,y) = \sum V_k(y) t^k/k!$ and $\hat{h}(t,y) = \sum h_k(y) t^k/k!$ and $\hat{\phi}(t,y) = \sum \phi_k(y) t^k/k!$ to the system (2.2.8)-(2.2.9),(2.2.6) satisfy $h_k\big|_{\partial\Omega} = 0$, for $k = 0, \dots, m$. However, since we are looking for solutions in Sobolev spaces this has to be expressed in a weak form:

$$h_k(y) \in H_0^1(\Omega), \qquad k = 0, ..., m. \tag{2.2.25}$$

We would like to think of (2.2.10)-(2.2.12) as determining h uniquely as a functional of V. In order for the initial value problem for the wave equation (2.2.10)-(2.2.12) to have a regular enough solution, the initial data needs to satisfy the compatibility conditions (2.2.25). These compatibility conditions for h will however depend on the formal power series for V. We now calculate the formal power series for the coupled system and hence the compatibility conditions. These power series are uniquely determined by x_0, V_0, h_0. By (2.2.8), using (2.2.20):

$$V_{k+1} = \textstyle\sum_{\ell \le k} S_{i\ell}^{jk}(\partial V_0, \dots, \partial V_{k-\ell-1}) \partial_j H_\ell, \qquad H_k = h_k + \phi_k. \tag{2.2.26}$$

Similarly by (2.2.9) we have for some function G_k:

$$e'(h_0) h_{k+1} = \textstyle\sum_{\ell \le k} S_{i\ell}^{jk}(\partial V_0, \dots, \partial V_{k-\ell-1}) \partial_j V_\ell + G_k(h_0, \dots, h_k). \tag{2.2.27}$$

The relation for ϕ_k is not as direct but it is clear from (2.2.23) that for some non local functional Φ_k:

$$\phi_k = \Phi_k[x_0, V_0, \dots, V_{k-1}, h_0, \dots, h_k]. \tag{2.2.28}$$

2.3 Tangential smoothing, tangential operators and tangential vector fields

There is a family of open sets V_μ, $\mu = 1, \ldots, N$ that cover $\partial\Omega$ and onto diffeomorphisms $\Phi_\mu : (-1,1)^2 \to V_\mu$. We fix a collection of cutoff functions $\chi_\mu : \partial\Omega \to \mathbb{R}$ so that χ_μ^2 form a partition of unity subordinate to the cover $\{V_\mu\}_{\mu=1}^N$, as well as another family of "fattened" cutoff functions $\widetilde{\chi}_\mu$ so that the support of $\widetilde{\chi}_\mu$ is contained in V_μ and so that $\widetilde{\chi}_\mu \equiv 1$ on the support of χ_μ. Recalling that Ω is the unit ball, we set $W_\mu = \{r\omega, r \in (1/2, 1], \omega \in V_\mu\}$ for $\mu = 1, \ldots, N$ and let W_0 be the ball of radius $3/4$ so that the collection $\{W_\mu\}_{\mu=0}^N$ covers Ω. Writing $\Psi_\mu(z, z_3) = z_3\Phi_\mu(z)$, Ψ_μ is a diffeomorphism from $(-1,1)^2 \times (1/2, 1]$ to W_μ. Let $\eta : [0,1] \to \mathbb{R}$ be a bump function so that $\eta(r) = 1$ when $1/2 \leq r \leq 1$ and $\eta(r) = 0$ when $r < 1/4$. We define cutoff functions on Ω by setting $\chi_\mu = \chi_\mu \eta$.

For a linear operator T' defined in coordinate charts we define a global operator T by

$$Tf = \sum T_\mu f, \quad \text{where} \quad T_\mu f = \chi_\mu \big(m_\mu^{-1} T' \big[m_\mu (\chi_\mu f) \circ \Psi_\mu \big] \big) \circ \Psi_\mu^{-1}, \tag{2.3.1}$$

where

$$m_\mu = |\det \Phi_\mu'|^{1/2} r. \tag{2.3.2}$$

Then T is symmetric with the measure dy if T' is with the measure dz is since $dS(\omega) = m_\mu^2 dz$.

2.3.1 Tangential smoothing

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be even, supported in $R = (-1,1)^2$ with $\int_{\mathbb{R}^2} \varphi = 1$ and let

$$T_\varepsilon f(z) = \int_{\mathbb{R}^2} \varphi_\varepsilon(z - z') f(z') dz', \qquad \text{where} \qquad \varphi_\varepsilon(z) = \varepsilon^{-2} \varphi(z/\varepsilon). \tag{2.3.3}$$

be a smoothing operator. Because φ is even, T_ε is symmetric; for any functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ we have:

$$\int T_\varepsilon f(z) g(z)\, dz = \int f(z) T_\varepsilon g(z)\, dz. \tag{2.3.4}$$

Furthermore, by Appendix A have:

$$|T_\varepsilon(fg)(z) - f T_\varepsilon(g)(z)| \leq C\varepsilon ||f||_{C^1(R)} ||g||_{L^2(R)}. \tag{2.3.5}$$

With notation as in (2.3.1), the smoothing operators we consider on Ω or $\partial\Omega$ are then defined by:

$$J_\varepsilon f = \sum\nolimits_{\mu=0}^{N} T_{\varepsilon,\mu} f, \qquad S_\varepsilon f = J_\varepsilon J_\varepsilon f = \sum\nolimits_{\mu,\nu=0}^{N} T_{\varepsilon,\nu} T_{\varepsilon,\mu} f. \tag{2.3.6}$$

Since T_ε is symmetric J_ε is as well. The following estimates are proved in Section 2.A.2:

Lemma 2.3.1. *With J_ε defined by (2.3.6), if $k \geq m$ then:*

$$||J_\varepsilon f||_{H^k(\partial\Omega)} \lesssim \varepsilon^{k-m} ||f||_{H^m(\partial\Omega)}, \quad ||J_\varepsilon f - f||_{H^k(\partial\Omega)} \lesssim \varepsilon ||f||_{H^{k+1}(\partial\Omega)}, \tag{2.3.7}$$

and, with $\Sigma = \partial\Omega$ or Ω:

$$||J_\varepsilon(fg) - f J_\varepsilon g||_{L^2(\Sigma)} \lesssim \varepsilon ||f||_{C^1(\Sigma)} ||g||_{L^2(\Sigma)}. \tag{2.3.8}$$

2.3.2 The tangential fractional derivatives

We will need to use fractional tangential derivatives to control our solution and we will define these operators in coordinates. If $F : \mathbb{R}^2 \to \mathbb{R}$, we define:

$$\langle \partial_\theta \rangle^s F(z) = \int_{\mathbb{R}^2} e^{iz\cdot\xi} \langle \xi \rangle^s \hat{F}(\xi)\, d\xi, \quad \text{where} \quad \hat{F}(\xi) = \int_{\mathbb{R}^2} e^{-iz\cdot\xi} F(z)\, dz, \tag{2.3.9}$$

and we define fractional tangential derivatives on Ω by:

$$\langle \partial_\theta \rangle^s_\mu f = \widetilde{\chi}_\mu (\langle \partial_\theta \rangle^s f_\mu) \circ \Psi_\mu^{-1}, \quad f_\mu = (\chi_\mu f) \circ \Phi, \qquad \mu = 1, ..., N. \tag{2.3.10}$$

We also set $\langle\partial_\theta\rangle_0^s f = \chi_0(\langle\partial\rangle^s f_0)\circ\Psi_0^{-1}$, where $\langle\partial\rangle^s$ is defined by taking the Fourier transform in all directions.

For $s \in \mathbb{R}$, $k \in \mathbb{N}$, we define:

$$||f||_{H^{(k,s)}(\Omega)} = \sum_{\mu=0}^{N} ||\langle\partial_\theta\rangle_\mu^s f||_{H^k(\Omega)}, \quad \text{and} \quad ||f||_{H^s(\partial\Omega)} = \sum_{\mu=1}^{N} ||\langle\partial_\theta\rangle_\mu^s f||_{L^2(\partial\Omega)}. \tag{2.3.11}$$

In Appendix A we prove:

Lemma 2.3.2. *If $T \in \mathcal{T}$, then:*

$$\Big|\int_{\partial\Omega} fTg\, dS(y)\Big| \le C||f||_{H^{1/2}(\partial\Omega)}||g||_{H^{1/2}(\partial\Omega)}, \tag{2.3.12}$$

$$\Big|\int_{\Omega} fTg\, dy\Big| \le C||f||_{H^{(0,1/2)}(\Omega)}||g||_{H^{(0,1/2)}(\Omega)}. \tag{2.3.13}$$

In addition, with $\Sigma = \partial\Omega$ or Ω,

$$||\langle\partial_\theta\rangle_\mu^{1/2}(fg) - f\langle\partial_\theta\rangle_\mu^{1/2} g||_{L^2(\Sigma)} \le C||f||_{H^2(\Sigma)}||g||_{L^2(\Sigma)}. \tag{2.3.14}$$

2.3.3 The tangential derivatives and tangential norms

Since Ω is the unit ball, the vector fields

$$\Omega_{ab} = y^a\partial_{y^b} - y^b\partial_{y^a}, \qquad a, b = 1, 2, 3, \tag{2.3.15}$$

are tangent to $\partial\Omega$ and span the tangent space there. With η the cutoff function defined above, we let:

$$\mathcal{T} = \cup_{a,b=1,2,3}\{\eta\,\Omega_{ab}, (1-\eta)\partial_{y^a}\}. \tag{2.3.16}$$

In analogy with the two dimensional case, when $\mathcal{T}$ is just the derivative with respect to the angle in polar coordinates, we will now introduce some simplified notation for the norms. Suppose that $\mathcal{T} = \{T_1, \dots, T_{N'}\}$. If $V : \Omega \to \mathbf{R}^3$ is a vector field we will let $\mathcal{T}V$ stand for the map $\mathcal{T}V:\Omega \to \mathbf{R}^{3N'}$, whose components are $T_j V^i$, for $i=1,2,3$, $j=1,...,N'$. Moreover let $\mathcal{T}^r = \mathcal{T}\times\cdots\times\mathcal{T}$ (r times) and let $T^I \in \mathcal{T}^r$ stand for a product of r vector fields in $\mathcal{T}$, where $I=(i_1,...,i_r)\in[1,N']\times\cdots\times[1,N']$ is a multiindex of length $|I|=r$. Let $\mathcal{T}^r V$ stand for the map $\mathcal{T}^r V:\Omega\to\mathbf{R}^{3N'r}$, whose components are $T^I V^i$, for $i=1,2,3$, $1\le i_j\le N'$, $j=1,...,r$. The norm of $\mathcal{T}^r V$ is

$$|\mathcal{T}^r V|^2 = \delta_{ij}\mathcal{T}^r V^i \cdot \mathcal{T}^r V^j, \quad \text{where} \quad \mathcal{T}^r V^i \cdot \mathcal{T}^r V^j = \sum_{|I|=r,\ T^I\in\mathcal{T}^r} T^I V^i\, T^I V^j. \tag{2.3.17}$$

We will use similar notation for space time vector fields tangential to the boundary. Let $\mathfrak{D}=\mathcal{T}\cup D_t$, and $\mathfrak{D}^r=\mathfrak{D}\times\cdots\times\mathfrak{D}$ (r times), $\mathfrak{D}^{r,k}=\mathcal{T}^r\times D_t^k$. For $K=(I,k)$ a multiindex with $|I|=r$, we write $D^K=T^I D_t^k$, $T^I\in\mathcal{T}^r$.

2.4 The smoothed Euler's equations

In this section, we introduce the smoothed problem we will use to construct solutions to (2.1.1)-(2.1.6). This in the incompressible case goes back to Coutand-Shkoller[6], with important improvements due to Nordgren[2].

2.4.1 Tangential smoothing of the coordinates

With the tangential smoothing operator S_ε defined as in (2.3.6), given a vector field V, we define the smoothed coordinate $\widetilde{x}(t,y)$ by:

$$\widetilde{x}^i(t,y) = x_0(y) + \int_0^t S_\varepsilon V^i(s,y)ds. \tag{2.4.1}$$

We then define:

$$A^i{}_a = \frac{\partial \widetilde{x}^i}{\partial y^a}, \qquad A^a_{\ i} = \left(A^{-1}\right)^a{}_i = \frac{\partial y^a}{\partial \widetilde{x}^i}. \tag{2.4.2}$$

We define $\widetilde{\mathcal{D}}_t = \widetilde{x}(t,\Omega)$ and we use the letters $i,j,k,...$ to denote coordinate derivatives $\widetilde{\partial}_i = \partial/\partial\widetilde{x}^i$ on $\widetilde{\mathcal{D}}_t$: However all our functions will be functions of y so we will think of $\widetilde{\partial}$ as a differential operator on Ω

$$\widetilde{\partial}_i f(t,y) = A^a{}_i(t,y)\partial_a f(t,y), \quad \text{where} \quad \partial_a = \partial/\partial y^a. \tag{2.4.3}$$

The coordinate $\widetilde{x}(t,y)$ and Euclidean metric on $\widetilde{\mathcal{D}}_t$ induces a time-dependent metric on Ω: $\widetilde{g}_{ab} = \delta_{ij}A^i{}_a A^j{}_b$. We let $\widetilde{\kappa}dy = \det(\partial\widetilde{x}/\partial y)dy$ be the volume element on Ω induced by the volume element $d\widetilde{x}$ on $\widetilde{\mathcal{D}}_t$. Let

$$\widetilde{\Delta} f = \delta^{ij}\widetilde{\partial}_i\widetilde{\partial}_j f = \widetilde{\kappa}^{-1}\partial_a\left(\widetilde{\kappa}\widetilde{g}^{ab}\partial_b f\right), \text{where} \widetilde{g}^{ab} = \delta^{ij}A^a{}_i A^b{}_j, \widetilde{\kappa} = \det(\partial\widetilde{x}/\partial y), \tag{2.4.4}$$

denote the Laplacian. Given a one-form $\alpha = \alpha_i d\widetilde{x}^i$ on $\widetilde{\mathcal{D}}_t$, we also write:

$$\operatorname{div}\alpha = \widetilde{\partial}_i(\delta^{ij}\alpha_j), \qquad \operatorname{curl}\alpha_{ij} = \widetilde{\partial}_i\alpha_j - \widetilde{\partial}_j\alpha_i. \tag{2.4.5}$$

Here, $a, b, c, \ldots$ correspond to quantities in the y variables and $i, j, k \ldots$ to quantities in the $\widetilde{x}$ variables.

2.4.2 The smoothed problem

Given initial data (V_0, h_0) which are compatible with (2.1.1)-(2.1.5) in the sense of (2.1.8), we now introduce the smoothed problem we will consider. Given a vector field $V : [0,T] \times \Omega \to \mathbb{R}^3$, we define the tangentially smoothed Lagrangian coordinate $\widetilde{x} = \widetilde{x}[V]$ by (2.4.1) and $A, \widetilde{\partial}$ and $\widetilde{\Delta}$ as in (2.4.2)-(2.4.4).

We would like to define $h = h[V]$ to be the unique solution to:

$$D_t^2 e(h) - \widetilde{\Delta} h + \rho(h) = (\widetilde{\partial}_i S_\varepsilon V^j)(\widetilde{\partial}_j V^i), \qquad \text{in } [0,T] \times \Omega, \tag{2.4.6}$$

$$h = 0, \text{ on } [0,T] \times \partial\Omega, \tag{2.4.7}$$

$$h(0,y) = h_0^\varepsilon(y), \qquad D_t h(0,y) = h_1^\varepsilon(y), \text{ in } \Omega, \tag{2.4.8}$$

for some choice of initial data $h_0^\varepsilon, h_1^\varepsilon$. However, there are compatibility conditions that must be satisfied in order for this to have a sufficiently regular solution. We will define these conditions momentarily but for now suppose that V and $h_0^\varepsilon, h_1^\varepsilon$ are such that this problem has a unique solution h. We then abuse notation slightly and write $\rho(t,y)$ instead of $\rho(h(t,y))$. We also define $\widetilde{\mathcal{D}}_t = \widetilde{x}(t,\Omega)$ and write $\widetilde{y}(t,\widetilde{x})$ for the inverse of the map $y \mapsto \widetilde{x}(t,y)$. Next, we define the gravitational potential $\phi = \phi[V]$ by:

$$\phi(t,y) = \frac{1}{4\pi}\int_{\widetilde{\mathcal{D}}_t} \frac{\rho(t,\widetilde{y}(t,x'))\,dx'}{|\widetilde{x}(t,y) - x'|} = \frac{1}{4\pi}\int_\Omega \frac{\rho(t,y')\widetilde{\kappa}(t,y')\,dy'}{|\widetilde{x}(t,y) - \widetilde{x}(t,y')|}, \tag{2.4.9}$$

so that

$$\widetilde{\Delta}\phi = -\rho(h) \tag{2.4.10}$$

Note that ϕ depends on V both because $\rho = \rho[V]$ and also through the domain $\widetilde{\mathcal{D}}_t$.

With the above definitions of $\widetilde{\partial}[V], h[V], \phi[V]$ in mind:

Definition 1. Given a vector field V, suppose that $h_0^\varepsilon, h_1^\varepsilon$ are given such that the the system (2.4.7)-(2.4.8) has a unique sufficiently regular solution $h = h[V]$. We say that V is a *solution to the smoothed problem* if:

$$D_t V^i = -\delta^{ij}\widetilde{\partial}_j h[V] - \delta^{ij}\widetilde{\partial}_j\phi[V], \qquad \text{in } [0,T]\times\Omega, \quad \text{and} \quad V^i(0,y) = V_0^i(y). \tag{2.4.11}$$

Note that subtracting the divergence of (2.4.11) from (2.4.7) using (2.4.9) gives that $D_t\big(D_t e(h) + \text{div}\, V\big) = 0$ so:

$$D_t e(h) + \text{div}\, V = 0, \qquad \text{in } [0,T]\times\Omega, \tag{2.4.12}$$

provided that this holds at $t = 0$.

In Section 2.12, we prove the following a priori estimate for the problem (2.4.11)-(2.4.12).

Let δ_0 denote the largest number so that (2.1.9) holds with $\delta = \delta_0$ at $t = 0$. Also set:

$$E_0^r = ||V_0||^2_{H^{(r-1,1/2)}(\Omega)} + ||x_0||^2_{H^r(\Omega)} + ||\widetilde{\partial} h_0||^2_{H^{r-1}(\Omega)} + \varepsilon^2(||V_0||^2_{H^r(\Omega)} + ||\widetilde{\partial} h_0||^2_{H^r(\Omega)}). \quad (2.4.13)$$

Writing $H^k = H^k(\Omega), H^{(k,1/2)} = H^{(k,1/2)}(\Omega)$, in Corollary 2.12.1, we prove:

Theorem 2.4.1. *Let $r \geq 8$ and fix ε sufficiently small. There are strictly positive, continuous functions $\mathscr{T}_r, \mathscr{C}_r$ with $\mathscr{T}_r$ independent of ε, and so that if $V \in C([0,T]; H^r(\Omega))$ solves the smoothed Euler equations (2.4.11)-(2.4.12) for $0 \leq t \leq T$ with $T \leq \mathscr{T}_r(E_0^r, 1/\delta_0)$, then, with $||\widetilde{\partial} h||_r = \sum_{k+\ell \leq r} ||D_t^k \widetilde{\partial} h||_{H^\ell}$:*

$$||V(t)||^2_{H^{(r-1,1/2)}} + ||\widetilde{x}(t)||^2_{H^r} + ||\widetilde{\partial} h(t)||^2_{r-1} + ||D_t^r h(t)||^2_{L^2} + \varepsilon^2\big(||V(t)||^2_{H^r} + ||\widetilde{\partial} h(t)||^2_r\big)$$

$$\leq \mathscr{C}_r(E_0^{r-1}, \delta_0^{-1}) E_0^r. \quad (2.4.14)$$

Before proving existence for the smoothed Euler equations (2.4.11) -(2.4.12), we need to ensure that given sufficiently regular V, the wave equation (2.4.7)-(2.4.8) has a unique sufficiently regular solution.

2.4.3 Compatibility conditions for the smoothed problem

We now define $h_0^\varepsilon, h_1^\varepsilon$ and give a condition that guarantees that the initial-boundary value problem (2.4.7)-(2.4.8) is well-posed.

We say that the initial data $V_0^\varepsilon, h_0^\varepsilon$ satisfy the compatibility conditions of order m if there is a formal power series solution $\hat{V}(t,y) = \sum V_k^\varepsilon(y) t^k/k!$, along with $\hat{h}(t,y) = \sum h_k^\varepsilon(y) t^k/k!$ and $\hat{\phi}(t,y) = \sum \phi_k^\varepsilon(y) t^k/k!$ which satisfy (2.4.11) and (2.4.12) at $t = 0$, and moreover so that:

$$h_k^\varepsilon \in H_0^1(\Omega), \quad k = 0, ..., m. \quad (2.4.15)$$

As in Section 2.2.2, repeatedly using that $[D_t, \widetilde{\partial}_j] = -(\widetilde{\partial}_j S_\varepsilon V^\ell)\widetilde{\partial}_\ell$, we have:

$$D_t^k \widetilde{\partial}_i = \widetilde{S}_{i\ell}^{jk} \widetilde{\partial}_j D_t^\ell, \tag{2.4.16}$$

where the sum is over $\ell \leq k$, $\widetilde{S}_{ik}^{jk} = \delta_i^j$ and for $\ell \leq k$, with the same constants $c_{\ell\ell_1 \ldots \ell_n}^{kn}$ as in (2.2.21), we have:

$$\widetilde{S}_{i\ell}^{jk} = \widetilde{S}_{i\ell}^{jk}(\widetilde{\partial}\widetilde{V}, ..., D_t^{k-\ell-1}\widetilde{V}) = c_{\ell\ell_1 \cdots \ell_n}^{kn} (\widetilde{\partial}_i D_t^{\ell_1} \widetilde{V}^{i_1}) \cdots (\widetilde{\partial}_{i_n} D_t^{\ell_n} \widetilde{V}^j), \tag{2.4.17}$$

where the sum is over $\ell_1 + \cdots + \ell_n = k - \ell - n$ and $n = 1, ..., k$, and where we are writing $\widetilde{V} = S_\varepsilon V$.

Using that $\hat{V}$ solves the smoothed-out Euler equations (2.4.11) at $t = 0$, the coefficients V_ℓ^ε must satisfy:

$$V_{k+1}^\varepsilon = \sum\nolimits_{\ell \leq k} \widetilde{S}_{i\ell}^{jk} (\widetilde{\partial}\widetilde{V}_0^\varepsilon, ..., \widetilde{\partial}\widetilde{V}_{k-\ell-1}^\varepsilon) \widetilde{\partial}_j H_\ell^\varepsilon, \tag{2.4.18}$$

with $H_\ell^\varepsilon = h_\ell^\varepsilon + \phi_\ell^\varepsilon$. Similarly, the condition that $\hat{h}$ solves the continuity equation (2.4.12) at $t = 0$ becomes:

$$e'(h_0^\varepsilon) h_{k+1}^\varepsilon = \sum\nolimits_{\ell \leq k} S_{i\ell}^{jk} (\widetilde{\partial} S_\varepsilon V_0^\varepsilon, ..., \widetilde{\partial} S_\varepsilon V_k^\varepsilon) + G_k(h_0^\varepsilon, ..., h_k^\varepsilon), \tag{2.4.19}$$

for a function M_k. We note the explicit formula for $k = 0$:

$$e'(h_0^\varepsilon) h_1^\varepsilon = -\operatorname{div} V_0^\varepsilon, \tag{2.4.20}$$

and we take this to be the definition of h_1^ε. In addition we have that there is a non-local function Φ_k so that:

$$\phi_k^\varepsilon = \Phi_k[x_0, \widetilde{V}_0^\varepsilon, ..., \widetilde{V}_{k-1}^\varepsilon, h_0^\varepsilon, ..., h_k^\varepsilon]. \tag{2.4.21}$$

To construct a solution to the smoothed Euler's equations (2.4.11), we will need to consider only vector fields V whose Taylor expansions in t at $t = 0$ agree with (2.4.18) and we make the following definition:

Definition 2. A vector field V is called *admissible* to order m if, for $k = 0, ..., m$:

$$D_t^k V|_{t=0} = \sum_{\ell \le k} S^{jk}_{i\ell}(\widetilde{\partial} S_\varepsilon V_0^\varepsilon, ...\widetilde{\partial} S_\varepsilon V^\varepsilon_{k-1})\widetilde{\partial}_j H^\varepsilon_\ell, \tag{2.4.22}$$

where the $\widetilde{S}^{jk}_{i\ell}$ are defined by (2.4.17) and where $H^\varepsilon_\ell = h^\varepsilon_\ell + \phi^\varepsilon_\ell$, with $V^\varepsilon_\ell, h^\varepsilon_\ell, \phi^\varepsilon_\ell$ defined by (2.4.18)- (2.4.21).

In other words, V is admissible to order m if it solves the smoothed Euler equations (2.4.11) to order m at $t = 0$.

In Theorem 2.F.1 we prove that if $(V_0^\varepsilon, h_0^\varepsilon)$ are compatible to order m in the sense of (2.4.15) and $V = V(t, y)$ is a fixed vector field satisfying (2.4.22) to order m, then the system (2.4.7)-(2.4.8) has a unique solution $h = h[V]$ on a time interval $[0, T]$ so that $h(t) \in H^1_0(\Omega)$ for $t \in [0, T]$ and so that $D_t^k h \in C([0, T]; H^{m-k}(\Omega))$ for $k = 0, ..., m$. By Theorem 2.E.1, given (V_0, h_0) which are compatible to order m, see (2.2.25), there is a function h_0^ε so that if V is admissible to order r, then (V_0, h_0^ε) are compatible to order r, see (2.4.15), and so that $h_0^\varepsilon \to h_0$ as $\varepsilon \to 0$.

2.4.4 Solving the smoothed problem

Suppose that (V_0, h_0) are given and are compatible in the sense of (2.1.8) (i.e. for the full nonlinear problem) to order r. In Appendix 2.E, we construct a sequence h_0^ε with $h_0^\varepsilon \to h_0$ as $\varepsilon \to 0$ and so that if V is any vector field satisfying (2.4.22) for $k = 1, ..., r$, then (V_0, h_0^ε) are compatible to order r in the sense of (2.4.15). Given initial data (V_0, h_0^ε) which are compatible to order r in the sense of (2.4.15) and an admissible vector field V, we define a functional:

$$\Lambda^i[V](t, y) = V_0^i(y) - \int_0^t \delta^{ij}\widetilde{\partial}_j h(s, y)\, ds - \int_0^t \delta^{ij}\widetilde{\partial}_j \phi(s, y)\, ds. \tag{2.4.23}$$

with $(\widetilde{\partial}, h, \phi) = (\widetilde{\partial}[V], h[V], \phi[V])$ as in the previous section. It is clear that if V is a fixed point of Λ then V is a solution of the smoothed problem. To construct a fixed point of Λ, we will use the following norms:

$$||V||_{\mathcal{X}^s(T)} = \sup_{0\leq t\leq T} ||V(t)||_{\mathcal{X}^s}, \tag{2.4.24}$$

where:

$$||V(t)||_{\mathcal{X}^s} = \sum_{k=1}^{s} ||D_t^k V(t)||_{H^{s-k}(\Omega)} + ||V(t)||_{H^{s-1}(\Omega)}. \tag{2.4.25}$$

Our first result is then:

Theorem 2.4.2. *Let $r \geq 7$, $\varepsilon > 0$ and suppose that $(V_0^\varepsilon, h_0^\varepsilon)$ are compatible to order r. Let $\mathscr{C}_r$ be as in Theorem 2.4.1 and set $\mathscr{C}_r' = \mathscr{C}_r E_0^r$. Then there is a positive continuous function $T_\varepsilon = T_\varepsilon(E_0^{r+1})$ so that for any $0 \leq T \leq T_\varepsilon$, the map Λ has a unique fixed point in the space:*

$$\mathcal{C}^r(T) = \big\{ V\colon [0,T]\times\Omega \to \mathbb{R}^3 \big|\, V \text{ satisfies (2.4.22) and } \sup_{0\leq t\leq T} ||V(t)||^2_{\mathcal{X}^{r+1}} \leq \varepsilon^{-2}\mathscr{C}_r' + 1 \big\}. \tag{2.4.26}$$

The rest of the paper is devoted to proving Theorems 2.4.1 and 2.4.2. We will see in our construction that $T_\varepsilon = O(\varepsilon)$ and in particular our proof of existence does not give a uniform time of existence as $\varepsilon \to 0$.

2.4.5 Existence up to an ε independent time

Combining the a priori estimate from Theorem 2.4.1 and the existence result Theorem 2.4.2, we have:

Proof of Theorem 2.1.1. Given initial data (V_0, h_0), define $(V_0^\varepsilon, h_0^\varepsilon)$ as in Appendix 2.E. For sufficiently small ε, let T_* denote the largest time so that the smoothed Euler equations (2.4.11) have a unique solution $V_\varepsilon(t) \in H^r(\Omega)$ with $V|_{t=0} = V_0^\varepsilon$. By Theorem 2.4.2, $T_* > 0$. We claim that in fact $T_* \geq \mathscr{T}_r$ where $\mathscr{T}_r$ is as in Theorem 2.4.1. Assuming that this holds for the moment, we now have a vector field $V_\varepsilon \in H^r(\Omega)$ satisfying (2.4.11) on a time interval $[0, T_E]$ independent of ε and moreover by the energy estimate (2.4.14) we have that $||V_\varepsilon||_{H^{r-1,1/2}(\Omega)}$ is uniformly bounded in ε. By standard compactness theorems, it follows that there is a vector field $V \in H^{(r-1,1/2)}(\Omega)$ so that $V_\varepsilon \to V$ strongly in $H^{r-1}(\Omega)$. Since $\widetilde{x} = S_\varepsilon x \to x$ as $\varepsilon \to 0$ and since $H^{r-1}(\Omega)$ is an algebra, it follows that V satisfies Euler's equations (2.1.1)-(2.1.2). To see that $T_* \geq \mathscr{T}_r$, we assume that $T_* < \mathscr{T}_r$. By the a priori estimate (2.4.14) and using that $D_t V_\varepsilon = -\widetilde{\partial} h - \widetilde{\partial}\phi$ and Theorem 2.7.4 to control ϕ, we have:

$$||V_\varepsilon(t)||^2_{\mathcal{X}^{r+1}} \leq ||V_\varepsilon(t)||^2_{H^r(\Omega)} + ||\widetilde{\partial} h(t)||_r^2 + ||\widetilde{\partial}\phi(t)||^2_r \leq \varepsilon^{-2}\mathscr{C}_r', \quad 0 \leq t \leq T. \tag{2.4.27}$$

Define $(V^\varepsilon_{T_*}, h^\varepsilon_{T_*}) = \lim_{t \nearrow T_*}(V(t), h(t))$. Since h solves the wave equation (2.4.7) and V_ε solves the smoothed-out Euler equations (2.4.11) it follows that the compatibility conditions (2.4.15) are satisfied at $t = T_*$ as well, so repeating the proof of Theorem 2.4.2 with t replaced by $t - T_*$ and $(V_0^\varepsilon, h_0^\varepsilon)$ replaced by $(V^\varepsilon_{T_*}, h^\varepsilon_{T_*})$, we see that there is a $T_2 > T_*$ so that $V \in \mathcal{C}^r(T_2)$ satisfies (2.4.7), which contradicts the fact that T_* was maximal. □

2.5 Elliptic estimates

In what follows we will need several elliptic estimates, which are modifications of the estimates from [10] and [2]. We summarize these here, and their proofs can be found in Appendix 2.B.

Let $V : [0, T] \times \Omega \to \mathbb{R}^3$ be a vector field on Ω and let $\widetilde{x}$ denote its smoothed flow as in (2.4.1), and let $A^i_{\ a}$ and $A^a_{\ i}$ be as in (2.4.2). We will assume that we have the following a priori bound:

$$\textstyle\sum_{i,a} |A^a_{\ i}| + |A^i_{\ a}| + \sum_{|I|\leq 3} |\partial_y^I \widetilde{x}| \leq M_0, \tag{2.5.1}$$

and in some of our estimates we will additionally assume the bound:

$$\textstyle\sum_{i,a} |A^{a}{}_{i}| + |A^{i}{}_{a}| + \sum_{k+|J|\le 3} |\partial_y^J \widetilde{x}| + |\partial_y^J D_t^k V| \le M. \tag{2.5.2}$$

We write $\widetilde{\partial}$ for the derivative with respect to $\widetilde{x}$ (as in (2.4.3)) and $\widetilde{\Delta}$ for the Laplacian with respect to $\widetilde{x}$. For a one-form $\alpha = \alpha_i d\widetilde{x}^i$ on $\widetilde{\mathcal{D}}_t$, we define $\operatorname{div}\alpha, \operatorname{curl}\alpha$ by (2.4.5). We will work with the following mixed norms:

$$||f||_{k,\ell} = \textstyle\sum_{s\le k} ||D_t^s f||_{H^\ell(\Omega)}, \qquad ||f||_r = \sum_{k+\ell\le r} ||f||_{k,\ell}. \tag{2.5.3}$$

In this section we let C_0, C_s for $s \ge 1$ and C_s' denote continuous functions of arguments indicated below

$$C_0 = C_0(M_0), \quad C_s = C_s(M_0, ||\widetilde{x}||_{H^s(\Omega)}), \quad C_s' = C_s'(M, ||\widetilde{x}||_s), \quad \text{for} \quad s \ge 1. \tag{2.5.4}$$

As in [11] and [2], we will rely on the following simple pointwise estimate:

Lemma 2.5.1. *With the norm of the tangential derivatives $\mathcal{T}\alpha$ as in (2.3.17), for every (0,1)-tensor α on Ω*

$$|\widetilde{\partial}\alpha| \le C_0\big(|\operatorname{div}\alpha| + |\operatorname{curl}\alpha| + |\mathcal{T}\alpha|\big). \tag{2.5.5}$$

See Lemma 2.B.1 for the proof. Then (2.5.5) can be used to prove (see Proposition 2.B.2):

Lemma 2.5.2. *Let $s = k + \ell \ge 1$. If α is a (0,1)-tensor on Ω then, with notation as in (2.3.17):*

$$||\alpha||_{H^s(\Omega)} \le C_s\big(||\operatorname{div}\alpha||_{H^{s-1}(\Omega)} + ||\operatorname{curl}\alpha||_{H^{s-1}(\Omega)} + \textstyle\sum_{j\le s} ||\mathcal{T}^j\alpha||_{L^2(\Omega)}\big), \tag{2.5.6}$$

$$||\alpha||_{k,\ell} \le C_s'\big(||\operatorname{div}\alpha||_{k,\ell-1} + ||\operatorname{curl}\alpha||_{k,\ell-1} + \textstyle\sum_{k_1\le k,\ell_1\le\ell} ||\mathfrak{D}^{k_1,\ell_1}\alpha||_{L^2(\Omega)}\big). \tag{2.5.7}$$

If $\alpha_i = \widetilde{\partial}_i f$ for a function f which vanishes on $\partial\Omega$, using an integration-by-parts argument to control the last term on the right-hand side of (2.5.6) (resp. (2.5.7)) gives (see Proposition

2.B.1):

Proposition 2.5.1. *If $f:\Omega\to\mathbb{R}$ is a function with $f=0$ on $\partial\Omega$ then, with $\mathcal{T}\widetilde{x}$ defined in* (2.3.17)*, for $k+\ell=s\geq 1$:*

$$||\widetilde{\partial}f||_{H^s(\Omega)}\leq C_s\big(||\widetilde{\Delta}f||_{H^{s-1}(\Omega)}+(||\mathcal{T}\widetilde{x}||_{H^s(\Omega)}+||\widetilde{x}||_{H^s(\Omega)})||f||_{L^2(\Omega)}\big), \tag{2.5.8}$$

$$||\widetilde{\partial}f||_{k,\ell}\leq C_s'\big(||\widetilde{\Delta}f||_{k,\ell-1}+(||D_t\widetilde{x}||_s+||\widetilde{x}||_s)||D_t^kf||_{L^2(\Omega)}\big). \tag{2.5.9}$$

There are two crucial points in the estimate (2.5.8). First, we are estimating $\widetilde{\partial}f$ in $H^s(\Omega)$ instead of f in $H^{s+1}(\Omega)$. For the proof of this we only need to commute the divergence with $s-1$ instead of s derivatives with the Laplacian, which would have generate terms with too many derivatives of $\widetilde{x}$. Moreover, by first applying (2.5.6), we can replace full y-derivatives of $\widetilde{\partial}f$ with tangential derivatives applied to $\widetilde{\partial}f$. This is why the right-hand side of (2.5.8) involves $||\mathcal{T}\widetilde{x}||_{H^s(\Omega)}$, which we can control more easily than $||\widetilde{x}||_{H^{s+1}(\Omega)}$.

We also use (2.5.6) to prove the following estimates. They show that one can control α in the interior by the divergence and curl of α and either the normal component of α on the boundary or the projection of α to the tangent space at the boundary. The first estimate will be used to control $||J_\varepsilon x||_{H^r(\Omega)}$ in terms of the energies that we define in Section 2.10 and the second will be used to control $||V||_{H^r(\Omega)}$.

Proposition 2.5.2. *Fix $r\geq 5$, $1\leq s\leq r$. If α is a vector field, then with notation as in* (2.3.17) *and $H^s=H^s(\Omega)$:*

$$||\alpha||_{H^s}^2\leq C_s\Big(||\mathrm{div}\,\alpha||_{H^{s-1}}^2+||\mathrm{curl}\,\alpha||_{H^{s-1}}^2+||\alpha||_{H^1}^2$$
$$+\sum_{\mu=1}^N\int_{\partial\Omega}(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{s-1}\alpha^i)\cdot(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{s-1}\alpha^j)N_iN_j dS\Big), \tag{2.5.10}$$

$$||\alpha||^2_{H^s} \leq C_s\Big(||\mathrm{div}\,\alpha||^2_{H^{s-1}}+||\mathrm{curl}\,\alpha||^2_{H^{s-1}}+||\alpha||^2_{H^1}$$

$$+\sum_{\mu=1}^N \int_{\partial\Omega} (\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{s-1}\alpha^i)\cdot(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{s-1}\alpha^j)\gamma_{ij}dS\Big). \quad (2.5.11)$$

Here γ denotes the projection to the tangent space at the boundary, for definition of $\langle\partial_\theta\rangle_\mu^{1/2}$, see Appendix 2.A.

2.5.1 Estimates for differences of solutions

In Section 2.9, we will prove that the map Λ defined in (2.4.23) satisfies a Lipschitz estimate. Given two vector fields $V_I, V_{II} : [0,T] \times \Omega \to \mathbb{R}$, define the corresponding smoothed flows $\widetilde{x}_I, \widetilde{x}_{II}$ as well as the derivatives $\widetilde{\partial}_I, \widetilde{\partial}_{II}$ and the Laplacians $\widetilde{\Delta}_I, \widetilde{\Delta}_{II}$. Assume that $\widetilde{x}_I, \widetilde{x}_{II}$ both satisfy the estimate (2.5.1) or (2.5.2) and now let the constants (2.5.4) depend on the corresponding norms of both $\widetilde{x}_I, \widetilde{x}_{II}$.

Proposition 2.5.3. *Fix $r \geq 6$. If $f, g : \Omega \to \mathbb{R}$ and $f = g = 0$ on $\partial\Omega$, then for $1 \leq \ell \leq r-1$, respectively $k+\ell = r$:*

$$||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{H^\ell(\Omega)} \leq C_r\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{H^{\ell-1}(\Omega)} + ||\widetilde{x}_I||_{H^r(\Omega)}||f-g||_{L^2(\Omega)}$$

$$+ ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)}||\widetilde{\partial}_{II} g||_{H^\ell(\Omega)}\big), \quad (2.5.12)$$

$$||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{H^r(\Omega)} \leq C_r\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{H^{r-1}(\Omega)} + ||\mathcal{T}\widetilde{x}_I||_{H^r(\Omega)}||f-g||_{L^2(\Omega)}$$

$$+ ||\mathcal{T}\widetilde{x}_I - \mathcal{T}\widetilde{x}_{II}||_{H^r(\Omega)}||\widetilde{\partial}_{II} g||_{H^r(\Omega)}\big), \quad (2.5.13)$$

$$||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{k,\ell} \le C_r' \big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{k,\ell-1} + \big(||\widetilde{x}_I||_r + ||D_t \widetilde{x}_I||_r\big) ||f-g||_{k,0}$$

$$+ \big(||\widetilde{x}_I - \widetilde{x}_{II}||_r + ||D_t(\widetilde{x}_I - \widetilde{x}_{II})||_r\big) ||\widetilde{\partial}_{II} g||_r\big). \quad (2.5.14)$$

2.6 Estimates for wave equations

As in the previous section, we fix a vector field $V = V(t,y)$ on Ω and let $\widetilde{x}(t,y)$ denote the tangentially smoothed flow of V. Define $A, \widetilde{\kappa}, \widetilde{\Delta}$ as in (2.4.2)-(2.4.4). We will assume that the a priori assumptions (2.5.2) hold Note that (2.5.2) combined with the formula for the derivative of the inverse (2.D.2) implies that $|\partial_y^\ell A^a_{\ i}| \le C(M)$ for $\ell \le 2$. We consider the initial-boundary value problem:

$$\sigma D_t^2 \varphi - \widetilde{\Delta}\varphi = \mathcal{F}, \quad \text{on } [0,T]\times\Omega, \quad \text{with} \quad \varphi = 0, \quad \text{on } [0,T]\times\partial\Omega, \quad (2.6.1)$$

$$\varphi(0,y) = \varphi_0(y), \qquad D_t\varphi(0,y) = \varphi_1(y), \quad \text{on } \Omega, \quad (2.6.2)$$

where $\mathcal{F}$ is a given function on $[0,T]\times\Omega$ and $\sigma = \sigma(\varphi)$ is a given function satisfying $0 < e_1 \le \sigma \le e_2$ for some e_1, e_2. We will suppress the dependence on e_1, e_2 in the following. In our applications we will have $\sigma = e'(\varphi)$ where $e(\varphi)$ is determined from the equation of state as in Section 2.2.1. We remark that for a linear equation of state $p(\rho) = \rho + c$, we have $e(\varphi) = \varphi + c$ and so in this case (2.6.1) is a linear wave equation.

For the applications we have in mind, we will need to allow $\mathcal{F}$ to depend on φ:

$$\mathcal{F}(t,y) = \mathcal{F}_1(t,y) + \mathcal{F}_2[\varphi], \quad (2.6.3)$$

where we assume that $\mathcal{F}_2$ satisfies the following estimates:

$$||D_t^s \mathcal{F}_2[\varphi]||_{L^2(\Omega)} \le P_1(||\varphi||_{s+1,0} + ||\varphi||_s), \qquad ||\mathcal{F}_2[\varphi]||_{s-1} \le P_2||\varphi||_s, \quad (2.6.4)$$

for some polynomials P_1, P_2 depending on $M, L, ||\widetilde{x}||_{H^s}, ||V||_{\mathcal{X}^s}, ||\varphi||_{s,0}, ||\varphi||_{s-1}$. Recall that the mixed norms $||\cdot||_{k,\ell}$ and $||\cdot||_s$ are defined in (2.5.3). In Section 2.8, we will take $F_2 = e''(\varphi)(D_t\varphi)^2 + \rho[\varphi]$ where ρ is determined from φ by the equation of state as in Section 2.2.1, and we will see that this satisfies (2.6.4).

The energy associated to the wave equation (2.6.1) is:

$$W_s(t) = \Big(\frac{1}{2}\sum_{k\leq s}\int_\Omega \big(\sigma(t)\,|D_t^{k+1}\varphi(t)|^2 + \delta^{ij}\big(D_t^k\widetilde{\partial}_i\varphi(t)\big)\big(D_t^k\widetilde{\partial}_j\varphi(t)\big)\big)\,\widetilde{\kappa}dy\Big)^{1/2}. \tag{2.6.5}$$

We assume that we have the following a priori estimate for φ:

$$\sum_{k+|J|\leq 3}|D_t^k\partial_y^J\widetilde{\partial}\varphi(t)| + |D_t^k\varphi(t)| \leq L, \quad \text{in } [0,T]\times\Omega. \tag{2.6.6}$$

The first goal of this section is to prove the following theorem:

Theorem 2.6.1. *Fix $s \geq 0$. There are continuous functions G_s, with*

$$G_s = G_s\big(M, L, T, W_{s-1}(0), \sup_{0\leq t\leq T}(||\widetilde{x}(t)||_{H^s(\Omega)} + ||V(t)||_{\mathcal{X}^s} + ||F_1(t)||_{s-2})\big), \tag{2.6.7}$$

so that if φ satisfies (2.6.1), (2.6.6) holds, and $\sup_{0\leq\tau\leq T}||\varphi(\tau)||_{s+1} < \infty$, then for $0\leq t\leq T$:

$$||\varphi(t)||_{s+1,0} + ||\widetilde{\partial}\varphi(t)||_{s,0} \leq G_s\Big(W_s(0) + \int_0^t ||F_1(\tau)||_{s,0} + ||F_1(\tau)||_{s-1} + ||V(\tau)||_{\mathcal{X}^{s+1}}\,d\tau\Big), \tag{2.6.8}$$

and

$$\begin{aligned} ||\widetilde{\partial}\varphi(t)||_s \leq G_s(||\mathcal{T}\widetilde{x}||_s + 1)\Big(&||F_1(t)||_{s-1} \\ &+ W_s(0) + \int_0^t ||F_1(\tau)||_{s,0} + ||F_1(\tau)||_{s-1} + ||V(\tau)||_{\mathcal{X}^{s+1}}\,d\tau\Big). \end{aligned} \tag{2.6.9}$$

In Appendix 2.F, we prove that if the compatibility conditions (2.F.4) hold to order s then the problem (2.6.1)-(2.6.2) has a unique solution on a time interval $[0,T]$ with $\sup_{0\leq\tau\leq T}||\varphi(\tau)||_{s+1} < \infty$. However if the compatibility conditions are not satisfied then these estimates need not be valid.

Proof. When $s = 0$, by Lemma 2.6.2 there is a continuous function $G_0' = G_0'(M)$ and a polynomial P_0 so that:

$$\frac{d}{dt} W_0 \leq G_0'\big(||F_1||_{L^2(\Omega)} + P_0(L, ||\varphi||_{L^2(\Omega)}) W_0 \big). \tag{2.6.10}$$

By Poincaré's inequality and Hölder's inequality, we have $||\varphi||_{L^2(\Omega)} \leq C(M) ||\widetilde{\partial}\varphi||_{L^2(\Omega)} \leq C(M)L$. Multiplying both sides of (2.6.10) by the integrating factor $e^{-(G_0' + P_0(L, C(M)L))t)}$ and integrating, we get:

$$W_0(t) \leq G_0 \Big(W_0(0) + \int_0^t ||F_1(\tau)||_{L^2(\Omega)}\, d\tau \Big), \tag{2.6.11}$$

for a continuous function $G_0 = G_0(M, L, T)$. We now assume that we have the result for $s = 0, ..., m-1$. Let G_m' be as in Lemma 2.6.2. By the inductive assumption, (2.6.19) gives:

$$\frac{d}{dt} W_m \leq G_m' \Big(\big(1 + P_m(L, W_{m-1}(0), \sup_{0 \leq t \leq T} ||F_1||_{m-2})\big) W_m$$

$$+ ||F_1||_{m,0} + ||F_1||_{m-1} + ||V||_{\mathcal{X}^m} \Big), \tag{2.6.12}$$

for a polynomial P_m, and so multiplying by the integrating factor $e^{-G_m'(1+P_m)t}$ and integrating gives the result for $s = m$ as well. The estimate (2.6.9) then follows from (2.6.8) and (2.6.13).

□

Before proving Lemma 2.6.2, we note the following consequence of the elliptic estimate (2.5.9):

Lemma 2.6.1. *There is a continuous functions $G_s'' = G_s''(M, ||\widetilde{x}||_s)$ and P_s so that if φ satisfies* (2.6.1), *then:*

$$||\widetilde{\partial}\varphi||_s \le G_s''(||\mathcal{T}\widetilde{x}||_{H^s} + ||V||_s)\big(||\varphi||_{s+1,0} + ||\widetilde{\partial}\varphi||_{s,0} + ||F||_{s-1}$$

$$+ P_s(L, ||\varphi||_{s,0}, ||\widetilde{\partial}\varphi||_{s-1,0}, ||F||_{s-2})\big). \quad (2.6.13)$$

Proof. For $s = 0$ there is nothing to prove and so we assume that (2.6.13) holds for $s = 0, 1, ..., n-1$. To prove that it holds for $s = n$, we will show that if $k + \ell = n$ then:

$$||D_t^k\widetilde{\partial}\varphi||_{H^\ell} \le G_n''\big(||\varphi||_{n+1,0} + ||\widetilde{\partial}\varphi||_{n,0} + ||F||_{n-1} + P(L, ||\varphi||_{n,0}, ||\widetilde{\partial}\varphi||_{n-1,0}, ||F||_{n-2})\big), \quad (2.6.14)$$

with $G_n'' = G_n''(M, ||\widetilde{x}||_n)$. There is nothing to prove if $\ell = 0$ and so we assume that this estimate holds for $\ell = 0, ..., \ell' - 1$. To prove that it holds for $\ell = \ell'$, we use the estimate (2.5.8) when $\ell' = n$:

$$||\widetilde{\partial}\varphi||_{H^n} \le C_n'\big(||\Delta\varphi||_{H^{n-1}} + (||\mathcal{T}\widetilde{x}||_{H^n} + ||\widetilde{x}||_{H^n})||\varphi||_{L^2}\big)$$

$$\le C_n'\big(||\sigma D_t^2\varphi||_{H^{\ell'-1}} + ||F||_{H^{\ell'-1}} + (||\mathcal{T}\widetilde{x}||_{H^n} + ||\widetilde{x}||_{H^n})||\varphi||_{L^2}\big), \quad (2.6.15)$$

and the estimate (2.5.9) when $n - \ell' \ge 1$:

$$||D_t^{n-\ell'}\widetilde{\partial}\varphi||_{H^{\ell'}(\Omega)} \le C_n'\big(||\Delta\varphi||_{n-\ell',\ell'-1} + (||D_t\widetilde{x}||_n + ||\widetilde{x}||_n)||D_t^{n-\ell'}\varphi||_{L^2(\Omega)}\big)$$

$$\le C_n'\big(||\sigma D_t^2\varphi||_{n-\ell',\ell'-1} + ||F||_{n-\ell',\ell'-1} + (||D_t\widetilde{x}||_n + ||\widetilde{x}||_n)||D_t^{n-\ell'}\varphi||_{L^2(\Omega)}\big), \quad (2.6.16)$$

Using (2.D.44), the first term here is bounded by $C||\varphi||_{n-\ell'+2,\ell'-1} + P(L, ||\varphi||_{n-1})$ and this second term can be bounded by the right-hand side of (2.6.14) by the inductive assumption. If $\ell' = 1$ then we have just proven (2.6.14). If $\ell' \ge 2$ we write $\partial\varphi/\partial y^a = A^i{}_a\widetilde{\partial}_i\varphi$ and use the product estimate (2.A.45) and Lemma 2.D.1:

$$||\varphi||_{n-\ell'+2,\ell'-1} \leq ||\partial_y \varphi||_{n-\ell'+2,\ell'-2} + ||\varphi||_{n-\ell'+2,\ell'-2}$$

$$\leq C(M, ||\widetilde{x}||_n)(||\widetilde{\partial}\varphi||_{n-\ell'+2,\ell'-2} + ||\varphi||_{n-1}), \quad (2.6.17)$$

and noting that $||\widetilde{x}||_n \leq C(||\widetilde{x}||_{H^n} + ||V||_{n-1})$, this implies (2.6.14). □

We then have the following energy estimate:

Lemma 2.6.2. *For each $s \geq 0$, there is a continuous function*

$$G_s'(t) = G_s'(M, ||\widetilde{x}(t)||_s, ||V(t)||_{\mathcal{X}^s}, W_{s-1}(t)) \quad (2.6.18)$$

and a polynomial P so that if φ satisfies (2.6.1), then:

$$\frac{d}{dt} W_s \leq G_s'\Big(W_s + ||F_1||_{s,0} + ||F_1||_{s-1} + ||V||_{\mathcal{X}^{s+1}} + P(L, W_{s-1}, ||F||_{s-2}) W_s \Big). \quad (2.6.19)$$

Proof. We start by showing:

$$\frac{d}{dt} W_s^2 \leq G_s''\Big(W_s + ||F||_{s,0} + ||\widetilde{\partial}\varphi||_s + ||V||_{\mathcal{X}^{s+1}} + P(L, ||\varphi||_s) W_s \Big) W_s, \quad (2.6.20)$$

for a continuous function $G_s'' = G_s''(M, ||\widetilde{x}||_s, ||V||_{\mathcal{X}^s})$. We have:

$$\frac{d}{dt} W_s^2 = \sum\nolimits_{k\leq s} \int_\Omega \sigma(D_t^{k+2}\varphi)(D_t^{k+1}\varphi) + \delta^{ij}(D_t^k \widetilde{\partial}_i \varphi)\widetilde{\partial}_j(D_t^{k+1}\varphi)\, \widetilde{\kappa} dy$$

$$+ \sum\nolimits_{k\leq s}\Big(\int_\Omega \delta^{ij}(D_t^k \widetilde{\partial}_i\varphi)[\widetilde{\partial}_j, D_t^{k+1}]\varphi\, \widetilde{\kappa} dy + \frac{1}{2}\int_\Omega (D_t\sigma)(D_t^{k+1}\varphi)^2$$

$$+ (D_t \log \widetilde{\kappa})\big((D_t^{k+1}\varphi)^2 + |D_t^k \widetilde{\partial}\varphi|^2\big)\, \widetilde{\kappa} dy \Big). \quad (2.6.21)$$

The last line is bounded by $C(M)(1+L)(W_s)^2$. Integrating by parts, the terms on the first line are:

$$\int_\Omega \big(\sigma D_t^{k+2}\varphi - \delta^{ij}(\widetilde{\partial}_j D_t^k \widetilde{\partial}_i\varphi)\big)(D_t^{k+1}\varphi)$$

$$= \int_\Omega \big(\sigma D_t^{k+2}\varphi - D_t^k\widetilde{\Delta}\varphi\big)(D_t^{k+1}\varphi) + \int_\Omega \delta^{ij}([D_t^k,\widetilde{\partial}_j]\widetilde{\partial}_i\varphi)(D_t^{k+1}\varphi)\,\widetilde{\kappa}dy. \quad (2.6.22)$$

By Lemma 2.D.7, we have:

$$||\sigma D_t^k(D_t^2\varphi) - D_t^k(\sigma D_t^2\varphi)||_{L^2(\Omega)} \leq P(L,||\varphi||_{k-1})||\varphi||_k, \quad (2.6.23)$$

and by the commutator estimate (2.D.34):

$$||[D_t^{k+1},\widetilde{\partial}_j]\varphi||_{L^2(\Omega)} \leq C_k(M,||\widetilde{x}||_k,||V||_{\mathcal{X}^k})\big(||\widetilde{\partial}f||_{k,0} + (||V||_{\mathcal{X}^{k+1}}+1)||\widetilde{\partial}f||_{k-1}\big), \quad (2.6.24)$$

$$||[D_t^k,\widetilde{\partial}_j]\widetilde{\partial}_i\varphi||_{L^2(\Omega)} \leq C_k(M,||\widetilde{x}||_k,||V||_{\mathcal{X}^k})(||\widetilde{\partial}^2f||_{k-1,0} + ||\widetilde{\partial}^2f||_{k-2}). \quad (2.6.25)$$

By (2.A.45), $||\widetilde{\partial}^2f||_{k-1,0} \leq C(M,||V||_k)(1+||V||_{\mathcal{X}^{k+1}})||\widetilde{\partial}f||_k$ and since $D_t^k(\sigma D_t^2\varphi) - D_t^k\widetilde{\Delta}\varphi = D_t^kF$, using (2.6.4) to control D_t^kF, we have (2.6.20). To prove (2.6.19) from (2.6.20), we now want to re-write $||\varphi||_s$ in terms of $||\varphi||_{s,0}$ and $||\widetilde{\partial}\varphi||_{s-1}$, and for this we re-write $\partial_a\varphi = A^i{}_a\widetilde{\partial}_i\varphi$ and use (2.A.45) and Lemma 2.D.1 to get: $||A^i{}_a\widetilde{\partial}_i\varphi||_{s-1} \leq C(M)||\widetilde{x}||_s||\widetilde{\partial}\varphi||_{s-1}$. This implies that $||\varphi||_s \leq C_s\big(||\varphi||_{s,0}+||\widetilde{\partial}\varphi||_{s-1}\big)$, and so inserting this into (2.6.20), applying (2.6.13) and bounding $||V||_s \leq ||V||_{\mathcal{X}^{s+1}}$ and using that $dW_s^2/dt = 2W_s dW_s/dt$ gives (2.6.19). $\square$

The following corollary will be used in Section 2.9:

Corollary 2.6.1. *Fix $r \geq 7$. Suppose that for some $T_1, K > 0$, we have the following estimate:*

$$\sup_{0\leq t\leq T_1}\big(||\widetilde{x}(t)||_r + ||V(t)||_{\mathcal{X}^{r+1}} + ||F_1(t)||_{r,0} + ||F_1(t)||_{r-1}\big) \leq K, \quad (2.6.26)$$

and that φ is a solution to (2.6.1) on $[0, T_1]$, satisfying:

$$\textstyle\sum_{k+|J|\le 3}|(D_t^k\partial_y^J\widetilde{\partial}\varphi)(0)| + |D_t^k\varphi(0)| \le L_0. \tag{2.6.27}$$

There are continuous functions Q_r and $\underline{G}_r = \underline{G}_r(M, L_0, W_5(0), K)$ so that if T satisfies:

$$TQ_r(M, L_0, W_5(0), K, T_1) \le 1, \quad \text{and } T \le T_1, \tag{2.6.28}$$

then for $0 \le t \le T$ the estimate (2.6.9) holds with G_r replaced with $\underline{G}_r$.

Proof. Let $L(t) = \sum_{|J|+k\le 3} |D_t^k\partial_y^J\widetilde{\partial}\varphi(t)| + |D_t^k\varphi(t)|$. By Sobolev embedding $L(t) \le C(||\widetilde{\partial}\varphi(t)||_5 + ||\varphi(t)||_5)$. Using the product estimate (2.A.45) we have $||\varphi||_5 \le C'(M, ||\widetilde{x}||_{H^7(\Omega)}, ||V||_{\mathcal{X}^7})(||\varphi||_{5,0} + ||\widetilde{\partial}\varphi||_4)$. Integrating once in time and then using the estimates (2.6.9) and (2.6.8), we have:

$$L(t) \le L(0) + C' \int_0^t ||\varphi(\tau)||_{6,0} + ||\widetilde{\partial}\varphi(\tau)||_6 \, d\tau \le L(0) + TP_0, \tag{2.6.29}$$

for a polynomial P_0 with $P_0 = P_0\big(M, L, W_5(0), \sup_{0\le t\le T} \big(||\widetilde{x}(t)||_{H^7(\Omega)} + ||V(t)||_{\mathcal{X}^7} + ||F_1(t)||_7\big)\big)$.

We take $T \le T_* \equiv \min(T_0, T_1)$ with T_0 defined by:

$$T_0P_0(M, 2L_0, W_5(0), K_1) \le L_0/2. \tag{2.6.30}$$

Let $S = \{0 \le t \le T_* : L(t) \le 2L_0\}$. Then S is nonempty, connected and closed. If $t \in S$ is an interior point then (2.6.29) and the fact that $t \le T_0$ shows that $t + \delta \in S$ for sufficiently small δ, so $L \le 2L_0$ for $t \le T_*$. The result now follows from Theorem 2.6.1 with $\underline{G}_r = G_r(M, 2L_0, T_r, W_{r-1}(0), K_1)$ and $Q_r = \max(1, (2L_0)^{-1})P_0$. □

2.6.1 Estimates for differences of solutions

We will also need to prove a Lipschitz estimate for Λ. We fix two vector fields V_I,V_{II} and, defining $\widetilde{x}_J, \widetilde{\partial}_J, \widetilde{\Delta}_J$ with $J = I, II$ as in (2.4.2)-(2.4.4), we consider solutions φ_J to

$$\sigma_J D_t^2 \varphi_J - \widetilde{\Delta}_J \varphi_J = \mathcal{F}_J, \quad \text{in } [0,T] \times \Omega, \tag{2.6.31}$$

$$\varphi_J = 0 \qquad \text{on } [0,T] \times \partial\Omega \tag{2.6.32}$$

for $J = I, II$, where $F_J = F_J^1 + F_J^2[\varphi_J]$ as in (2.6.3) with the same initial data as (2.6.2). We assume that F_J^2 satisfies (2.6.4) and:

$$||D_t^s (F_I^2[\varphi_I] - F_{II}^2[\varphi_{II}])||_{L^2} \leq P_1 ||\varphi_I - \varphi_{II}||_{s+1,0}, \tag{2.6.33}$$

$$||F_I^2[\varphi_I] - F_{II}^2[\varphi_{II}]||_{s-1} \leq P_2 (||\varphi_I - \varphi_{II}||_{s,0} + ||\widetilde{\partial}_I \varphi_I - \widetilde{\partial}_{II} \varphi_{II}||_{s-1}), \tag{2.6.34}$$

where $L^2 = L^2(\Omega)$, P_1, P_2 depend on M and $||\widetilde{x}_J||_{H^s(\Omega)}, ||V_J||_{\mathcal{X}^s}, ||\widetilde{\partial}\varphi_J||_s$, $J = I, II$. We will also assume that

$$\textstyle\sum_{k+|M|\leq 3} |D_t^k \partial_y^M \widetilde{\partial} \varphi_J| + |D_t^k \varphi_J| \leq L, \quad \text{in } [0,T] \times \Omega, \quad \text{for} \quad J = I, II. \tag{2.6.35}$$

Writing $\psi = \varphi_I - \varphi_{II}$, we have that:

$$\sigma_I D_t^2 \psi - \widetilde{\Delta}_I \psi = \mathcal{F}_I - \mathcal{F}_{II} + (\widetilde{\Delta}_I - \widetilde{\Delta}_{II}) \varphi_{II} + (\sigma_I - \sigma_{II}) D_t^2 \varphi_{II}, \quad \text{in } [0,T] \times \Omega, \tag{2.6.36}$$

$$\psi = 0, \qquad \text{on } [0,T] \times \partial\Omega, \tag{2.6.37}$$

and that $\psi|_{t=0} = D_t \psi|_{t=0} = 0$. In Lemma 2.6.3 we prove estimates for ψ that are similar to the estimates in Theorem 2.6.1 for φ_I, φ_{II}. However because of the terms $(\widetilde{\Delta}_I - \widetilde{\Delta}_{II})\varphi_{II}$ and $(\sigma_I - \sigma_{II}) D_t^2 \varphi_{II}$ we will need to assume an estimate for one more derivative of φ_{II} than we get back for ψ.

With notation as in the beginning of this section, we assume that both (u, V_I) and (w, V_{II}) satisfy the a priori assumption (2.5.2), and we also assume that φ_I, φ_{II} satisfy (2.6.35). We define:

$$||\alpha||_{C^k_{x,t}(\Omega)} = \sum_{k_1+k_2\leq k}||\partial_y^{k_1} D_t^{k_2}\alpha||_{L^\infty(\Omega)}. \tag{2.6.38}$$

We then have the following estimate:

Lemma 2.6.3. *Suppose that* $(A_I, V_I), (A_{II}, V_{II})$ *satisfy* (2.5.2) *and that* φ_I, φ_{II} *satisfy the wave equation* (2.6.1)-(2.6.2) *with* $\widetilde{\Delta}$ *replaced with* $\widetilde{\Delta}_I, \widetilde{\Delta}_{II}$ *and* $\mathcal{F}_2$ *(defined by* (2.6.3)*) replaced with* $\mathcal{F}_2[\varphi_I], \mathcal{F}_2[\varphi_{II}]$*, respectively. Define:*

$$W_s^{I,II}(t) = \Big(\frac{1}{2}\sum_{k\leq s}\int_\Omega e'(\varphi_I)|D_t^{k+1}(\varphi_I - \varphi_{II})|^2 + |D_t^k\widetilde{\partial}_I(\varphi_I - \varphi_{II})|^2\,\widetilde{\kappa}dy\Big)^{1/2}. \tag{2.6.39}$$

For each $s \geq 0$*, there is a positive, continuous function* D_s *depending on*

$$M, L, T, W_{s-1}(0),\ \text{and}\ \ \sup_{0\leq t\leq T}\big(||\widetilde{x}_J(t)||_{H^{s+1}} + ||V_J(t)||_{\mathcal{X}^{s+2}} + ||F_J(t)||_{s+1}\big), \text{for} J = I, II, \tag{2.6.40}$$

so that:

$$W_s^{I,II}(t) \leq D_s\int_0^t ||F_I(\tau) - F_{II}(\tau)||_{s,0} + ||F_I(\tau) - F_{II}(\tau)||_{s-1}$$
$$+ ||V_I(\tau) - V_{II}(\tau)||_{s+1} + ||\widetilde{x}_I(\tau) - \widetilde{x}_{II}(\tau)||_{C^3_{x,t}}d\tau, \tag{2.6.41}$$

and

$$||\widetilde{\partial}_I\varphi_I - \widetilde{\partial}_I\varphi_{II}||_s \leq D_s\big((||\mathcal{T}(x_I - x_{II})||_{H^s(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^3_{x,t}}$$
$$+ 1) + ||F_I - F_{II}||_{s-1} + W_s^{I,II}\big). \tag{2.6.42}$$

Proof. We will show that:

$$\frac{d}{dt}W_s^{I,II} \leq D_s'\Big(||F_I - F_{II}||_{s,0} + ||F_I - F_{II}||_{s-1} + W_s^{I,II}$$

$$+ ||V_I - V_{II}||_{\mathcal{X}^{s+1}}\big(||\varphi_{II}||_{s+2,0} + ||\widetilde{\partial}_{II}\varphi_{II}||_{s,1} + ||\varphi_{II}||_{s+1,0} + ||\widetilde{\partial}_{II}\varphi_{II}||_s\big)\Big), \quad (2.6.43)$$

where D_s' depends on $M, L, W_s^{I,II}(t)$ as well as $||\widetilde{\partial}_J\varphi_J||_s, ||\varphi_J||_{s+1,0}$ for $J = I, II$. Arguing as in the proof of Theorem 2.6.1 and using (2.6.8) and (2.6.9), this implies (2.6.41). Using(2.6.32) and (2.6.2), it just remains to prove that the L^2 norms of $D_t^s(\widetilde{\Delta}_I - \widetilde{\Delta}_{II})\varphi_{II}$ and $D_t^s((\sigma_I - \sigma_{II})D_t^2\varphi_{II})$ are bounded by (2.6.43). These terms are the reason that we lose derivatives of φ_{II} relative to ψ and why the coefficients D_s will depend on $||V_I||_{\mathcal{X}^{s+2}}, ||V_{II}||_{\mathcal{X}^{s+2}}$.

We start by controlling $D_t^s(\widetilde{\Delta}_I - \widetilde{\Delta}_{II})\varphi_{II}$ in L^2. We write:

$$D_t^s(\widetilde{\partial}_{Ii}\widetilde{\partial}_{Ij} - \widetilde{\partial}_{IIi}\widetilde{\partial}_{IIi})\varphi_{II} = (\widetilde{\partial}_{Ii}D_t^s\widetilde{\partial}_{Ij} - \widetilde{\partial}_{IIi}D_t^s\widetilde{\partial}_{IIi})\varphi_{II} + ([\widetilde{\partial}_{Ii}, D_t^s]\widetilde{\partial}_{Ij} - [\widetilde{\partial}_{IIi}, D_t^s]\widetilde{\partial}_{IIi})\varphi_{II}. \quad (2.6.44)$$

The first term is bounded in $L^2(\Omega)$ by $C(M)||(\widetilde{\partial}_I - \widetilde{\partial}_{II})\varphi_{II}||_{s,1}$. We write $(\widetilde{\partial}_I - \widetilde{\partial}_{II})\varphi_{II} = (u - w)\cdot\widetilde{\partial}_{II}\varphi_{II} + \widetilde{\partial}_{II}\varphi_{II}$, and by the product rule (2.A.45) this term is bounded by $C(M, ||V_I||_{\mathcal{X}^{s+1}}, ||V_{II}||_{\mathcal{X}^{s+1}})||V_I - V_{II}||_{\mathcal{X}^{s+1}}||\widetilde{\partial}_{II}\varphi_{II}||_{s+1}$. Using the commutator estimate (2.D.23), the second term has L^2 norm bounded by the right-hand side of (2.6.43).

To control $D_t^s(\sigma_I - \sigma_{II})D_t^2\varphi_{II}$, we use (2.A.45):

$$||D_t^s((\sigma_I - \sigma_{II})D_t^2\varphi_{II})||_{L^2(\Omega)} \leq D_s''||\varphi_I - \varphi_{II}||_{s,0}||\varphi_{II}||_{s+2,0}, \quad (2.6.45)$$

where D_s'' depends on L, and $||\varphi_J||_{s-1}$ for $J = I, II$.

The estimate (2.6.42) follows in the same way as (2.6.9) using the elliptic estimate (2.B.25) in place of (2.5.9) and:

$$\widetilde{\Delta}_I\varphi_I - \widetilde{\Delta}_{II}\varphi_{II} = \sigma_I D_t^2\psi + (\sigma_I - \sigma_{II})D_t^2\varphi_{II} + \mathcal{F}_I - \mathcal{F}_{II}. \qquad \square$$

2.7 Estimates for the gravitational potential

The estimates for ϕ will require integration by parts on $\widetilde{\mathcal{D}}_t \equiv \widetilde{x}(t,\Omega)$ and this will yield a boundary term which is difficult to deal with because there is no boundary condition for ϕ on $\partial\widetilde{\mathcal{D}}_t$. Because of this, following [9], our strategy is to extend the domain $\widetilde{\mathcal{D}}_t$ in the radial direction to a set $\widehat{\mathcal{D}}_t$ (see (2.7.6)), and then approximate ϕ with a sequence of functions ϕ_m defined $\widehat{\mathcal{D}}_t$ in such a way that $\mathcal{T}\Delta\phi_m$ and $D_t\Delta\phi_m$ vanish outside of $\widetilde{\mathcal{D}}_t$. We will also see that $\Phi(x-z) \in L^2_z$ if $x \in \partial\widehat{\mathcal{D}}_t$ and $z \in \widehat{\mathcal{D}}_t$ and these facts will allow us to bound $||\mathcal{T}^s\partial\phi_m||_{L^2(\widehat{\mathcal{D}}_t)}$ for each m after integrating by parts. We will also show that the sequence $\{\mathcal{T}^s\partial\phi_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2(\widehat{\mathcal{D}}_t)$, which gives an estimate for $||\mathcal{T}^s\partial\phi||_{L^2(\widetilde{\mathcal{D}}_t)}$ by letting $m \to \infty$. Finally, we get an estimate for $||\phi||_{H^{s+1}(\widetilde{\mathcal{D}}_t)}$ using (2.5.6).

Similarly to Section 2.5, we will let C_s, C_s', C_s'', C_s''' denote continuous functions with:

$$C_s = C_s(M_0, ||\widetilde{x}||_{H^s}), \quad C_s' = C_s'(M, ||\widetilde{x}||_s), \tag{2.7.1}$$

$$C_s'' = C_s''(M_0, ||\widetilde{x}||_{H^{(s-1,1/2)}}), \quad C_s''' = C_s'''(M, ||\widetilde{x}||_{H^{(s-1,1/2)}}, ||\widetilde{x}||_s), \tag{2.7.2}$$

with $H^k = H^k(\Omega)$ and $H^{(k,1/2)} = H^{(k,1/2)}(\Omega)$ defined by (2.3.11).

2.7.1 Bounds for ϕ and the extended domain $\widehat{\Omega}$

The following theorem is the main result of this section, and follows from the elliptic estimate (2.5.6) and the upcoming Theorem 2.7.3.

Theorem 2.7.1. *If $r \geq 5$, then with ϕ defined by* (2.4.9)*:*

$$||\widetilde{\partial}\phi||_{H^{r-1}(\widetilde{\mathcal{D}}_t)} \leq C_r\big(||\rho||_{H^{r-2}(\widetilde{\mathcal{D}}_t)} + ||\rho||_{H^{(r-2,1/2)}(\widetilde{\mathcal{D}}_t)}\big), \tag{2.7.3}$$

$$||\widetilde{\partial}\phi||_{H^{r}(\widetilde{\mathcal{D}}_t)} \leq C_r(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1)\big(||\rho||_{H^{r-1}(\widetilde{\mathcal{D}}_t)} + ||\rho||_{H^{(r-1,1/2)}(\widetilde{\mathcal{D}}_t)}\big). \tag{2.7.4}$$

We now employ the strategy mentioned above. Fix $d_0 > 0$ and define $\Omega^{d_0} = \{y_1 + y_2 \in \mathbb{R}^3 : y_1 \in \Omega, |y_2| < d_0\}$. Let χ_m be a smooth radial function whose support is contained in $\Omega^{d_0/2}$ with $\chi_m(y) = 1$ whenever $y \in \Omega$, where m is taken large enough that $1/m \leq d_0/2$. For fixed $r \geq 7$, let E denote an extension operator which is bounded from $H^r(\Omega)$ to $H^r(\Omega^{d_0})$ (see Appendix 2.A.4 for the detailed construction of the extension operator E). Define $\widehat{x}(t,y) = \chi_{d_0/2}(y)E(\widetilde{x}(t,\cdot) - x_0(\cdot))(y) + x_0(y)$, and define the corresponding velocity by $\widehat{V} = D_t\widehat{x}$. With these definitions, we have arranged that $\widehat{x}(t,y) = x_0(y)$ for $y \in \partial\Omega^{d_0}$, and for some $0 < c < C < \infty$:

$$c||\widetilde{x}||_{H^s(\Omega)} \leq ||\widehat{x}||_{H^s(\Omega^{d_0})} \leq C||\widetilde{x}||_{H^s(\Omega)}, \qquad ||\mathcal{T}\widehat{x}||_{H^s(\Omega^{d_0})} \leq C||\mathcal{T}\widetilde{x}||_{H^s(\Omega)}, \leq r, \tag{2.7.5}$$

for $0 \leq s \leq r$, by Theorem 2.A.1 and similarly for $\widehat{V}$. Abusing notation, we will also let χ_m denote the analogous function in the Eulerian frame and write $\chi_m(\widehat{x}) = \chi_m(\widehat{x}(t,y))$. We use $\widehat{\partial}$ to denote the derivative with respect to $\widehat{x}$ and $D_t = \partial_t + \widehat{V}^k\widehat{\partial}_k$ to denote the material derivative in $\widehat{\mathcal{D}}_t$.

Assuming that (2.5.1) holds, then taking d_0 smaller if necessary, $\widehat{x}(t,\cdot)$ is a homeomorphism from Ω^{d_0} to $\widehat{\mathcal{D}}_t$ and the normal N to $\partial\widetilde{\mathcal{D}}_t$ can be extended continuously into the region between $\partial\widetilde{\mathcal{D}}_t$ and $\partial\widehat{\mathcal{D}}_t$, where:

$$\widehat{\mathcal{D}}_t = \widehat{x}(t, \Omega^{d_0}). \tag{2.7.6}$$

We want to establish an approximation scheme which allows us to control ϕ. Let Φ be the fundamental solution of the Laplacian and let $\widehat{\rho}(x) = \rho(x)$ if $x \in \widetilde{\mathcal{D}}_t$, $\widehat{\rho}(x) = \bar{\rho}$ if $x \notin \widetilde{\mathcal{D}}_t$. We define

$$\phi_m(t,x) = -\widehat{\rho}\chi_m * \Phi(x), \qquad x \in \widehat{\mathcal{D}}_t. \tag{2.7.7}$$

We will show that the sequence $\{\mathcal{T}^j\widehat{\partial}\phi_m\}_{m=0}^{\infty}$ is Cauchy in $L^2(\widehat{\mathcal{D}}_t)$, which we will use to control $||\mathcal{T}^j\widetilde{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)}$. The fundamental result we need is the following inequality, whose proof can be found in Section 2.C.1:

Theorem 2.7.2. *Fix $r \geq 5$, suppose that (2.5.1) holds and let Φ denote the fundamental solution of the Laplacian in $\mathbb{R}^3$. If g is a smooth function supported in $\widehat{x}(t, \Omega^{d_0/2})$ such that $g(x)$ is radial when $x \in \widehat{\mathcal{D}}_t \backslash \mathcal{D}_t$, then:*

$$\sum_{j \leq r} ||\mathcal{T}^j \widehat{\partial}(g * \Phi)||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r''(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1)$$

$$\cdot \Big(\sum_{k \leq r-1} ||\mathcal{T}^k g||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathcal{T}^{r-1} g||_{H^{(0,1/2)}(\widehat{\mathcal{D}}_t)} + \sum_{k \leq 2} ||\mathcal{T}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)} \Big). \quad (2.7.8)$$

Applying (2.7.8) to $g = \widehat{\rho}(\chi_m - \chi_n)$ and using that by construction $\mathcal{T}(\widehat{\rho}(\chi_m - \chi_n)) = 0$, we have the following:

Corollary 2.7.1. *With the same hypotheses and notation as in Theorem 2.7.2 and with ϕ_ℓ defined by (2.7.7), set $\phi_{m,n} = \phi_m - \phi_n$ and $\chi_{m,n} = \chi_m - \chi_n$. Then*

$$\sum_{j \leq r-1} ||\mathcal{T}^j \widehat{\partial} \phi_{m,n}||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r''\big(||\widehat{\rho}\chi_{m,n}||_{L^2(\widehat{\mathcal{D}}_t)} + ||\widehat{\rho}\chi_{m,n}||_{L^6(\widehat{\mathcal{D}}_t)} + ||\widehat{\rho}\chi_{m,n}||_{L^\infty(\widehat{\mathcal{D}}_t)}\big), \quad (2.7.9)$$

$$||\mathcal{T}^r \widehat{\partial} \phi_{m,n}||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r''\big(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1\big)\big(||\widehat{\rho}\chi_{m,n}||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$+ ||\widehat{\rho}\chi_{m,n}||_{L^6(\widehat{\mathcal{D}}_t)} + ||\widehat{\rho}\chi_{m,n}||_{L^\infty(\widehat{\mathcal{D}}_t)}\big). \quad (2.7.10)$$

Corollary 2.7.1 implies that the sequence $\{\mathcal{T}^j \widehat{\partial} \phi_m\}_{m=1}^\infty$ is Cauchy in $L^2(\widehat{\mathcal{D}}_t)$ so $\mathcal{T}^j \widehat{\partial} \phi_m \to \mathcal{T}^j \widehat{\partial} \phi$ in $L^2(\widehat{\mathcal{D}}_t)$. This allows us to get a bound for $||\mathcal{T}^j \widetilde{\partial} \phi||_{L^2(\widetilde{\mathcal{D}}_t)}$ from that of $||\mathcal{T}^j \widehat{\partial} \phi_m||_{L^2(\widehat{\mathcal{D}}_t)}$. Although $g = \widehat{\rho}\chi_m$ is not smooth, by a regularization procedure in the radial and tangential directions (2.7.8) still holds.

Theorem 2.7.3. *With the same hypotheses and notation as in Theorem 2.7.2, writing $\phi = -\rho\chi_{\mathcal{D}_t} * \Phi$,* *we have:*

$$\textstyle\sum_{j\le r-1}||\mathcal{T}^j\widetilde{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)} \le C_r''\big(||\rho||_{H^{r-2}(\widetilde{\mathcal{D}}_t)} + ||\rho||_{H^{(r-2,1/2)}(\widetilde{\mathcal{D}}_t)}\big), \tag{2.7.11}$$

$$||\mathcal{T}^r\widetilde{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)} \le C_r''\big(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1\big)\big(||\rho||_{H^{r-1}(\widetilde{\mathcal{D}}_t)} + ||\rho||_{H^{(r-1,1/2)}(\widetilde{\mathcal{D}}_t)}\big). \tag{2.7.12}$$

Proof. Substituting $g = -\widehat{\rho}\chi_m$ into (2.7.8), we have:

$$\begin{aligned}\textstyle\sum_{j\le r}||\mathcal{T}^j\widehat{\partial}\phi_m||_{L^2(\widehat{\mathcal{D}}_t)} &\le C_r''(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1)\\ &\cdot\Big(\textstyle\sum_{k\le r-1}||\mathcal{T}^k(\widehat{\rho}\chi_m)||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathcal{T}^{r-1}(\widehat{\rho}\chi_m)||_{H^{(0,1/2)}(\widehat{\mathcal{D}}_t)} +\\ &\textstyle\sum_{k\le 2}||\mathcal{T}^k(\widehat{\rho}\chi_m)||_{L^6(\widehat{\mathcal{D}}_t)} + ||\widehat{\rho}\chi_m||_{L^\infty(\widehat{\mathcal{D}}_t)}\Big).\end{aligned} \tag{2.7.13}$$

Since the right hand side involves only tangential derivatives and because $\widehat{\rho}\chi_m \to \rho\chi_{\widetilde{\mathcal{D}}_t}$ as $m\to\infty$, we have that $||\mathcal{T}^{r-1}(\widehat{\rho}\chi_m)||_{H^{(0,0.5)}(\widehat{\mathcal{D}}_t)} \to ||\mathcal{T}^{r-1}\rho||_{H^{(0,0.5)}(\widetilde{\mathcal{D}}_t)}$ and $||\mathcal{T}^k(\widehat{\rho}\chi_m)||_{L^2(\widehat{\mathcal{D}}_t)} \to ||\mathcal{T}^k\rho||_{L^2(\widetilde{\mathcal{D}}_t)}$ for $k \le r-1$, and these are both bounded by the right-hand side of (2.7.11) (resp. (2.7.12)). Similarly, for $k \le r-1$ we have $||\mathcal{T}^k(\widehat{\rho}\chi_m)||_{L^6(\widehat{\mathcal{D}}_t)} \to ||\mathcal{T}^k\rho||_{L^6(\widetilde{\mathcal{D}}_t)}$ and by the Sobolev lemma, this last term is bounded by the right-hand side of (2.7.11) (resp. (2.7.12)). The term involving the L^∞ norm can be bounded in the same way. □

2.7.2 Bounds for ϕ with mixed space and time derivatives

The purpose of this section is to estimate $||D_t^{k-1}\widetilde{\partial}\phi||_{H^\ell(\widehat{\mathcal{D}}_t)}$, extending the result of Theorem 2.7.1. Recall the notation from (2.7.2).

Theorem 2.7.4. *Fix $r \ge 7$, $k \ge 1$ and suppose that (2.5.2) holds. Then with ϕ defined by (2.4.9) we have*

$$||D_t^{k-1}\widetilde{\partial}\phi||_{H^\ell(\widetilde{\mathcal{D}}_t)} \le C_r'\textstyle\sum_{s\le k-1}||D_t^s\rho||_{H^{r-s-1}(\widetilde{\mathcal{D}}_t)}, \qquad \textit{if} \quad k+\ell \le r, \tag{2.7.14}$$

$$||D_t^{k-1}\widetilde{\partial}\phi||_{H^\ell(\widetilde{\mathcal{D}}_t)} \leq C_r'(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)}+1)\sum_{s\leq k-1}||D_t^s\rho||_{H^{r-s}(\widetilde{\mathcal{D}}_t)}, \qquad \textit{if} \quad k+\ell=r. \tag{2.7.15}$$

In addition, C_r' *in* (2.7.14)-(2.7.15) *can be replaced by* $C(||\widetilde{x}||_{H^r(\Omega)}, \sum_{s\leq k-1}||D_t^sV||_{H^{r-s}(\Omega)})$.

Proof. We will just prove (2.7.15), the proof of (2.7.14) being similar. We proceed by induction: when $k+\ell=1$, this follows from Theorem 2.7.1. Suppose that we know (2.7.15) for $k+\ell=1,\cdots r$. The case $\ell=0$ follows directly from Theorem 2.7.2 so we assume that $\ell\geq 1$. By the elliptic estimate (2.5.9), we have:

$$||D_t^{k-1}\widetilde{\partial}\phi||_{H^\ell(\widetilde{\mathcal{D}}_t)} \leq C(||\widetilde{x}||_{H^r(\Omega)})\big(||\operatorname{div} D_t^{k-1}\widetilde{\partial}\phi||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)}+||\operatorname{curl} D_t^{k-1}\widetilde{\partial}\phi||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)}$$

$$+\sum_{s\leq\ell}||\mathcal{T}^sD_t^{k-1}\widetilde{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)}\big). \tag{2.7.16}$$

To control $||\operatorname{div} D_t^{k-1}\widetilde{\partial}\phi||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)}$ and $||\operatorname{curl} D_t^{k-1}\widetilde{\partial}\phi||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)}$, we use (2.D.39) and get:

$$||\operatorname{div} D_t^{k-1}\widetilde{\partial}\phi||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)} \leq ||D_t^{k-1}\rho||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)}$$

$$+P(\sum_{s\leq k-2}||D_t^sS_\varepsilon V||_{H^{r-s}(\Omega)})\sum_{s\leq k-2}||D_t^s\widetilde{\partial}\phi||_{H^{r-1-s}(\widetilde{\mathcal{D}}_t)}, \tag{2.7.17}$$

$$||\operatorname{curl} D_t^{k-1}\widetilde{\partial}\phi||_{H^{\ell-1}(\widetilde{\mathcal{D}}_t)} \leq P(\sum_{s\leq k-2}||D_t^sS_\varepsilon V||_{H^{r-s}(\Omega)})\sum_{s\leq k-2}||D_t^s\widetilde{\partial}\phi||_{H^{r-1-s}(\widetilde{\mathcal{D}}_t)}. \tag{2.7.18}$$

By the inductive assumption, $||D_t^s\widetilde{\partial}\phi||_{H^{r-s}(\widetilde{\mathcal{D}}_t)}$ is bounded by the right-hand side of (2.7.14) (resp. (2.7.15)) when $s\leq k-2$, and by Theorem 2.7.2, we likewise control $||\mathcal{T}^sD_t^{k-1}\widetilde{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)}$ for $s\leq\ell$. $\square$

First, we need a result analogous to Theorem 2.7.2. Let $\mathfrak{D}^r$ be the mixed tangential space and time derivative defined in Section 2.3.3. The proof of the following theorem can be found in Appendix 2.C.2:

Theorem 2.7.5. *Fix $r \geq 5$, suppose that (2.5.1) holds and let Φ denote the fundamental solution of the Laplacian in $\mathbb{R}^3$. If g is a smooth function whose support is contained in $\widehat{x}(t, \Omega^{d_0/2})$ which additionally satisfies that $g(x)$ is radial whenever $x \in \widehat{\mathcal{D}}_t \backslash \mathcal{D}_t$, then with $L^p = L^p(\widehat{\mathcal{D}}_t)$:*

$$\begin{aligned}\textstyle\sum_{k\leq r} ||\mathfrak{D}^k \widehat{\partial}(g * \Phi)||_{L^2} \\ \leq C_r'''(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1)\big(\textstyle\sum_{k\leq r} ||\mathfrak{D}^k g||_{L^2} + \sum_{k\leq 2} ||\mathfrak{D}^k g||_{L^6} + ||g||_{L^\infty}\big).\end{aligned} \tag{2.7.19}$$

Similar to the case when $\mathfrak{D}^j = \mathcal{T}^j$, Theorem 2.7.5 with $g = \widehat{\rho}(\chi_m - \chi_n)$ implies that the sequence $(\mathfrak{D}^j \widehat{\partial} \widehat{\phi}_m)_{m=1}^\infty$ is Cauchy in $L^2(\widehat{\mathcal{D}}_t)$ for $j \leq r$ and this gives the following bound:

Corollary 2.7.2. *With the same hypotheses and notation as in Theorem 2.7.5, writing $\phi = -\rho\chi_{\mathcal{D}_t} * \Phi$, we have:*

$$\textstyle\sum_{j\leq r-1} ||\mathfrak{D}^j \widehat{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)} \leq C_r'''\big(\sum_{k\leq r-1} ||\mathfrak{D}^k \rho||_{L^2(\widetilde{\mathcal{D}}_t)} + \sum_{k\leq 2} ||D_t^k \rho||_{H^{3-k}(\widetilde{\mathcal{D}}_t)}\big), \tag{2.7.20}$$

$$\begin{aligned}\textstyle\sum_{j\leq r} ||\mathfrak{D}^j \widehat{\partial}\phi||_{L^2(\widetilde{\mathcal{D}}_t)} \\ \leq C_r'''(||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)} + 1)\big(\textstyle\sum_{k\leq r} ||\mathfrak{D}^k \rho||_{L^2(\widetilde{\mathcal{D}}_t)} + \sum_{k\leq 2} ||D_t^k \rho||_{H^{3-k}(\widetilde{\mathcal{D}}_t)}\big).\end{aligned} \tag{2.7.21}$$

2.7.3 Fractional derivative bounds for ϕ

We will need an estimate for $||\widetilde{\partial}\phi||_{H^{(0,r-1/2)}(\mathcal{D}_t)}$ in Section 2.12. The following theorem is an analogue of Theorem 2.7.3 and follows from an approximation argument as above and the estimates in Appendix 2.A. See Appendix 2.C for the proof.

Theorem 2.7.6. *Fix $r \geq 5$. With ϕ defined by* (2.4.9) *we have*

$$||\mathcal{T}^{r-1}\widetilde{\partial}\phi||_{H^{(0,1/2)}(\widetilde{\mathcal{D}}_t)} \leq C_r||\rho||_{H^{r-1}(\widetilde{\mathcal{D}}_t)}. \tag{2.7.22}$$

2.7.4 Estimates for differences of solutions

Let $V_I, V_{II} : \Omega \to \mathbb{R}^3$ be two vector fields and $\widetilde{x}_I$, $\widetilde{x}_{II}$ their corresponding smoothed flows. Let $\widehat{x}_I$, $\widehat{x}_{II}$ be the corresponding flow maps in the extended domain Ω^{d_0} and $\widehat{V}_I = D_t\widehat{x}_I$, $\widehat{V}_{II} = D_t\widehat{x}_{II}$ be the associated velocity fields. For $J = I, II$, we define $\widetilde{\mathcal{D}}_{Jt} = \widetilde{x}_J(t,\Omega)$ and:

$$\phi_J(t,x) = \int_{\widetilde{\mathcal{D}}_{Jt}} \rho_J(y_J(t,z))\chi_{\widetilde{\mathcal{D}}_{Jt}}(z)\Phi(x-z)\,dz, \tag{2.7.23}$$

where $y_J(t,\cdot) : \widetilde{\mathcal{D}}_{Jt} \to \Omega$ is the inverse of $\widetilde{x}_J(t,\cdot)$. Throughout this section let D_r denote a continuous function depending on:

$$||\widetilde{x}_I||_{H^r(\Omega)}, ||\widetilde{x}_{II}||_{H^r(\Omega)}, ||V_I||_{H^{r-1}(\Omega)}, ||V_{II}||_{H^{r-1}(\Omega)}, ||D_tV_I||_{r-1}, ||D_tV_{II}||_{r-1}. \tag{2.7.24}$$

To prove a Lipschitz estimate for the map Λ in Section 2.9 we will use:

Theorem 2.7.7. *For $r \geq 7$, if $k+\ell = r$ then with ϕ_J defined by* (2.7.23)*:*

$$\begin{aligned}||D_t^{k-1}\widetilde{\partial}_I\phi_I - D_t^{k-1}\widetilde{\partial}_{II}\phi_{II}||_{H^\ell(\Omega)} \leq D_r\big(||\rho_I - \rho_{II}||_{r-1} \\ + \{||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} + ||V_I - V_{II}||_{\mathcal{X}^r}\}||\rho_{II}||_{r-1}\big).\end{aligned} \tag{2.7.25}$$

In addition, D_r in (2.7.14)-(2.7.15) *can be replaced by a continuous function $D_{k,\ell}$ depending on:*

$$||\widetilde{x}_I||_{H^r(\Omega)}, ||\widetilde{x}_{II}||_{H^r(\Omega)}, \textstyle\sum_{s\leq k-1}||D_t^sV_I||_{H^{r-s}(\Omega)}, \sum_{s\leq k-1}||D_t^sV_{II}||_{H^{r-s}(\Omega)}.$$

Proof. First, if $k=1$, by Lemma 2.B.2, we have:

$$||\widetilde{\partial}_I\phi_I - \widetilde{\partial}_{II}\phi_{II}||_{H^{r-1}(\Omega)} \leq D_r\big(||\rho_I - \rho_{II}||_{H^{r-2}(\Omega)}$$

$$+ ||\mathcal{T}^{r-1}(\widetilde{\partial}_I\phi_I - \widetilde{\partial}_{II}\phi_{II})||_{L^2(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)}||\widetilde{\partial}_{II}\phi_{II}||_{H^{r-1}(\Omega)}\big), \quad (2.7.26)$$

which is bounded by the right-hand side of (2.7.25). Second, we consider the case when $k \geq 2$. If $\ell = 0$ then (2.7.25) is Theorem 2.7.9. When $\ell \geq 1$, we set $\alpha = \widetilde{\partial}_I\phi_I$ and $\beta = \widetilde{\partial}_{II}\phi_{II}$ in Lemma 2.B.2 and get with $H^k = H^k(\Omega)$:

$$||D_t^{k-1}\widetilde{\partial}_I\phi_I - D_t^{k-1}\widetilde{\partial}_{II}\phi_{II}||_{H^\ell}$$

$$\leq D_r\Big(||\mathrm{div}_I D_t^{k-1}\widetilde{\partial}_I\phi_I - \mathrm{div}_{II} D_t^{k-1}\widetilde{\partial}_{II}\phi_{II}||_{H^{\ell-1}} + ||\mathrm{curl}_I D_t^{k-1}\widetilde{\partial}_I\phi_I - \mathrm{curl}_{II} D_t^{k-1}\widetilde{\partial}_{II}\phi_{II}||_{H^{\ell-1}}$$

$$+ ||\mathcal{T}^\ell D_t^{k-1}(\widetilde{\partial}_I\phi_I - \widetilde{\partial}_{II}\phi_{II})||_{L^2} + ||\widetilde{\partial}_I\phi_I - \widetilde{\partial}_{II}\phi_{II}||_{H^{r-1}} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r}||D_t^{k-1}\widetilde{\partial}_{II}\phi_{II}||_{H^\ell}\Big), \quad (2.7.27)$$

which is bounded by the right-hand side of (2.7.25) except for the first two terms. For the first term, we write:

$$\mathrm{div}_I\, D_t^{k-1}\widetilde{\partial}_I\phi_I - \mathrm{div}_{II}\, D_t^{k-1}\widetilde{\partial}_{II}\phi_{II} = D_t^{k-1}(\rho_I - \rho_{II})$$

$$+ \sum \big((D_t^{k_1}\partial\widetilde{x}_I)\cdots(D_t^{k_s}\partial\widetilde{x}_I)(\widetilde{\partial}_I D_t^{\ell'}\widetilde{\partial}_I\phi_I)\big) - \big((D_t^{k_1}\partial\widetilde{x}_{II})\cdots(D_t^{k_s}\partial\widetilde{x}_{II})(\widetilde{\partial}_{II} D_t^{\ell'}\widetilde{\partial}_{II}\phi_{II})\big), \quad (2.7.28)$$

with the sum taken over all $k_1 + \cdots + k_s + \ell' = k-1$ and $k_1 \geq 1$. The $H^{\ell-1}(\Omega)$ norm is controlled by

$$C_r\Big(||\rho_I - \rho_{II}||_{k-1,\ell-1} + \sum_{s\le k-2} ||D_t^s(\widetilde{\partial}_I\phi_I - \widetilde{\partial}_{II}\phi_{II})||_{H^{r-2-s}}$$

$$+ \big\{||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r} + ||V_I - V_{II}||_{\mathcal{X}^r}\big\}||D_t^s\widetilde{\partial}_{II}\phi_{II}||_{H^{r-2-s}}\Big), \quad (2.7.29)$$

by adapting the argument used in the proof of Lemma 2.C.6. The curl term is controlled similarly. □

With $f_J = (g_J * \Phi)\circ \widehat{x}_J$, $J = I, II$, the following theorem allows one to control $||\mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}$. This will be used to get an estimate for $||\mathfrak{D}^{r-1}\widetilde{\partial}_I\phi_I - \mathfrak{D}^{r-1}\widetilde{\partial}_{II}\phi_{II}||_{L^2(\Omega)}$ and by Proposition 2.B.2 this will allow us to control the full Sobolev norm of the difference. The proof of this is in Appendix 2.C.

Theorem 2.7.8. *For $r \ge 7$ there is a continuous D_r as in (2.7.24) so that the following hold. For $J = I, II$, if g_J are smooth functions supported in $\Omega^{d_0/2}$, such that $\mathfrak{D} g_J = 0$ in $\Omega^{d_0}\backslash\Omega$, then $f_J = (g_J * \Phi)\circ \widehat{x}_J$ satisfy:*

$$\sum_{k\le r-1} ||\mathfrak{D}^k\widehat{\partial}_I f_I - \mathfrak{D}^k\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}$$

$$\le D_r\Big(\sum_{k\le r-1}||\mathfrak{D}^k(g_I - g_{II})||_{L^2(\Omega^{d_0})} + \sum_{k\le 2}||\mathfrak{D}^k(g_I - g_{II})||_{L^6(\Omega^{d_0})} + ||g_I - g_{II}||_{L^\infty(\Omega^{d_0})}$$

$$+ \big\{||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} + ||V_I - V_{II}||_r\big\}\big(\sum_{k\le r-1}||\mathfrak{D}^k g_{II}||_{L^2(\Omega^{d_0})}$$

$$+ \sum_{k\le 2}||\mathfrak{D}^k g_{II}||_{L^6(\Omega^{d_0})} + ||g_{II}||_{L^\infty(\Omega^{d_0})}\big)\Big). \quad (2.7.30)$$

Let ϕ_I^m, ϕ_{II}^m be the extended ϕ_I and ϕ_{II}, respectively, i.e., $\phi_I^m = \int_{\Omega^{d_0}} [\widehat{\rho}_I\chi_m](\widehat{z}_I(t,y'))\Phi(\widehat{x}_I(t,y) - \widehat{z}_I(t,y'))\,dy'$ and ϕ_{II}^m is defined in a analogous way. Then Theorem 2.7.8 with $F = \phi_I^m - \phi_I^n$ and $G = \phi_{II}^m - \phi_{II}^n$ implies that the sequence $(\mathfrak{D}^k\widehat{\partial}_I\phi_I^m - \mathfrak{D}^k\widehat{\partial}_{II}\phi_I^m)_{m=1}^\infty$ is Cauchy in $L^2(\Omega^{d_0})$, and this allows one to get a bound for $||\mathfrak{D}^r\widetilde{\partial}_I\phi_I - \mathfrak{D}^r\widetilde{\partial}_{II}\phi_{II}||_{L^2(\Omega)}$ from that of $||\mathfrak{D}^r\widehat{\partial}_I\phi_I^m - \mathfrak{D}^r\widehat{\partial}_{II}\phi_{II}^m||_{L^2(\Omega^{d_0})}$, which gives:

Theorem 2.7.9. *If $r \geq 7$, there is a continuous D_r depending on the quantities in (2.7.24) so that with ϕ_I, ϕ_{II} defined by (2.7.23):*

$$\sum_{k\leq r-1}||\mathfrak{D}^k\widetilde{\partial}_I\phi_I - \mathfrak{D}^k\widetilde{\partial}_{II}\phi_{II}||_{L^2(\Omega)}$$
$$\leq D_r\Big(\sum_{k\leq r-1}||\mathfrak{D}^k(\rho_I-\rho_{II})||_{L^2(\Omega)} + \sum_{k\leq 2}||D_t^k(\rho_I-\rho_{II})||_{H^{3-k}(\Omega)}$$
$$+\big\{||\widetilde{x}_I-\widetilde{x}_{II}||_{H^r(\Omega)} + |V_I-V_{II}||_r\big\}\big(\sum_{k\leq r-1}||\mathfrak{D}^k\rho_{II}||_{L^2(\Omega)} + \sum_{k\leq 2}||D_t^k\rho_{II}||_{H^{3-k}(\Omega)}\big)\Big). \tag{2.7.31}$$

2.8 Estimates for solutions of the enthalpy equation

With the same notation as in the previous sections, we now return to the equation:

$$e'(h)D_t^2h - \widetilde{\Delta}h = (\widetilde{\partial}_iS_\varepsilon V^j)(\widetilde{\partial}_jV^i) - e''(h)(D_th)^2 - \rho(h), \quad \text{in } [0,T]\times\Omega, \tag{2.8.1}$$

$$h = 0, \qquad \text{on } [0,T]\times\partial\Omega, \tag{2.8.2}$$

$$h(0,y) = h_0^\varepsilon(y), \quad D_th(0,y) = h_1^\varepsilon(y), \qquad \text{on } \Omega. \tag{2.8.3}$$

We set:

$$\mathcal{W}_s(t) = \Big(\frac{1}{2}\sum_{k\leq s}\int_\Omega e'(h)|D_t^{k+1}h(t)|^2 + \delta^{ij}\big(D_t^k\widetilde{\partial}_ih(t)\big)\big(D_t^k\widetilde{\partial}_jh(t)\big)\widetilde{\kappa}(t)\,dy\Big)^{1/2}. \tag{2.8.4}$$

By Lemma 2.D.9, writing $\mathcal{F}_1 = -(\widetilde{\partial}_iS_\varepsilon V^j)(\widetilde{\partial}_jV^i)$ and $\mathcal{F}_2 = -e'(h)(D_th)^2 - \rho(h)$, we have the estimates:

$$||\mathcal{F}_1||_{s,0} = \sum_{k\leq s} ||D_t^k \mathcal{F}_1||_{L^2} \leq C(M)\big(||D_t^s V||_{H^1} + P(||V||_{\mathcal{X}^s})\big), \tag{2.8.5}$$

$$||\mathcal{F}_1||_{s-1} = \sum_{k+\ell\leq s-1} ||D_t^k \mathcal{F}_1||_{H^\ell} \leq C(M)\big(||V||_s + ||\widetilde{x}||_{H^s} + P(||V||_{s-1}, ||\widetilde{x}||_{H^{s-1}})\big), \tag{2.8.6}$$

and assuming that h satisfies the a priori assumption (2.6.6) we have

$$||\mathcal{F}_2[h]||_{s,0} \leq P_1(L, ||h||_{s,0}, ||h||_{s-1})||h||_{s+1,0}, \quad ||\mathcal{F}_2[h]||_{s-1} \leq P_2(L, ||h||_{s-1})||h||_s. \tag{2.8.7}$$

Combining these estimates with Theorem 2.6.1, we have:

Proposition 2.8.1. *Fix $r \geq 7$, $0 \leq s \leq r$, and $T > 0$. Suppose that $V \in \mathcal{X}^{r+1}(T)$ and that (2.5.2) holds. There is a continuous function $\overline{\mathcal{C}}_s$ depending on $M, L, \mathcal{W}_{s-1}(0)$ and $\sup_{0\leq t\leq T}\big(||V(t)||_{\mathcal{X}^s} + ||\widetilde{x}(t)||_{H^s}\big)$ so that if h satisfies the wave equation (2.8.2)-(2.8.3) and the a priori assumption (2.6.6) for $0 \leq t \leq T$, then for $s \leq r-1$:*

$$||D_t h(t)||_{s,0} + ||\widetilde{\partial} h(t)||_{s,0} \leq \overline{\mathcal{C}}_s\Big(\mathcal{W}_s(0) + \int_0^t ||V(\tau)||_{\mathcal{X}^{s+1}}\, d\tau\Big), \quad 0 \leq t \leq T, \tag{2.8.8}$$

$$||\widetilde{\partial} h(t)||_s \leq \overline{\mathcal{C}}_s\Big(||V(t)||_s + ||\widetilde{x}(t)||_{H^s} + \mathcal{W}_s(0) + \int_0^t ||V(\tau)||_{\mathcal{X}^{s+1}}\, d\tau\Big), \tag{2.8.9}$$

$$||\widetilde{\partial} h(t)||_r \leq \overline{\mathcal{C}}_r\Big(||V(t)||_r + \varepsilon^{-1}||J_\varepsilon x(t)||_{H^r} + \mathcal{W}_r(0) + \int_0^t ||V(\tau)||_{\mathcal{X}^{r+1}}\Big). \tag{2.8.10}$$

Moreover, with h_k^ε defined as in Section 2.4.3, suppose that:

$$\sum_{k+|J|\leq 3} |\partial_y^J \widetilde{\partial} h_k^\varepsilon| + |h_k^\varepsilon| \leq L_0. \tag{2.8.11}$$

If T satisfies (2.6.28) then the constants $\mathcal{C}_s$ can be taken to depend on L_0 instead of L if $T \leq T_1$.

2.8.1 Estimates for differences of solutions

We now prove the estimates we will need in Corollary 2.9.2. Recall the notation and definitions from Section 2.6. Suppose that h_J, for $J = I, II$, satisfy:

$$e'(h_J)D_t^2 h_J - \widetilde{\Delta}_J h_J = (\widetilde{\partial}_{Ji} S_\varepsilon V_J^j)(\widetilde{\partial}_{Jj} V_J^i) - e''(h_J)(D_t h_J)^2 - \rho(h_J), \quad \text{in } [0,T]\times\Omega, \tag{2.8.12}$$

$$h_J = 0, \quad \text{on } [0,T]\times\partial\Omega, \qquad h_J(0,y) = h_0^\varepsilon(y), \quad D_t h_J(0,y) = h_1^\varepsilon(y), \quad \text{on } \Omega. \tag{2.8.13}$$

We write $\mathcal{W}_s^I, \mathcal{W}_s^{II}$ for the energy (2.8.4) evaluated at h_I, h_{II}, respectively, and $F_J^1 = -(\widetilde{\partial}_{Ji} S_\varepsilon V_J^j)(\widetilde{\partial}_{Jj} V_J^i)$ for $J = I, II$. By the estimate (2.D.58) we have with $\mathcal{C}_s = \mathcal{C}_s(M, ||V_I||_s, ||V_{II}||_s, ||\widetilde{x}_I||_{H^{\ell+1}}, ||\widetilde{x}_{II}||_{H^{\ell+1}})$:

$$||F_I^1 - F_{II}^1||_{s,0} \leq \mathcal{C}_s\big(||D_t^s(V_I - V_{II})||_{H^1(\Omega)} + ||V_I - V_{II}||_{C^3_{x,t}}\big), \tag{2.8.14}$$

$$||F_I^1 - F_{II}^1||_{s-1} \leq \mathcal{C}_s\big(||V_I - V_{II}||_{\mathcal{X}^s} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^s} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^4_{x,t}}\big). \tag{2.8.15}$$

Writing $\mathcal{F}_J^2 = -e''(h_J)(D_t h_J)^2 - \rho(h_J)$, by the estimates (2.D.59)-(2.D.60), we also have:

$$||F_I^2 - F_{II}^2||_{s,0} \leq \mathcal{C}_s\big(||h_I - h_{II}||_{s+1,0} + ||V_I - V_{II}||_{C^3_{x,t}}\big), \tag{2.8.16}$$

$$||F_I^2 - F_{II}^2||_{s-1} \leq \mathcal{C}_s\big(||V_I - V_{II}||_{\mathcal{X}^s} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^s} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^4_{x,t}}\big). \tag{2.8.17}$$

Combining these estimates with Lemma 2.6.3, we have:

Corollary 2.8.1. *Define:*

$$\mathcal{W}_s^{I,II}(t) = \Big(\frac{1}{2}\sum_{k\leq s}\int_\Omega e'(h_I)|D_t^{k+1}(h_I - h_{II})|^2 + |D_t^k \widetilde{\partial}_I(h_I - h_{II})|^2\,\widetilde{\kappa}_I dy\Big)^{1/2}. \tag{2.8.18}$$

Fix $r \geq 7$ and suppose that the hypotheses in Proposition 2.6.1 hold. Take T small enough that (2.6.28) holds. For each $s \leq r-1$ there is a positive continuous function $\mathcal{D}_s$ depending on $M, L_0, \mathcal{W}_s(0)$, and

$\sup_{0\le t\le T}\big(||V_J(t)||_{\mathcal{X}^{r+1}} + ||\widetilde{x}_J(t)||_{H^r}\big)$, *for* $J = I, II$, *so that:*

$$\sup{}_{0\le t\le T} W_s^{I,II}(t) \le$$

$$\mathcal{D}_s \int_0^T ||V_I(\tau) - V_{II}(\tau)||_{\mathcal{X}^{r+1}} + ||\widetilde{x}_I(\tau) - \widetilde{x}_{II}(\tau)||_{H^r(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^4_{x,t}(\Omega)}\, d\tau, \quad (2.8.19)$$

$$||\widetilde{\partial}_I h_I - \widetilde{\partial}_{II} h_{II}||_s \le \mathcal{D}_s\big(W_s^{I,II} + ||V_I - V_{II}||_{s+1} + ||x_I - x_{II}||_{H^s}\big). \quad (2.8.20)$$

2.9 Existence for the smoothed problem up to a smoothing dependent time

Let $(V_0^\varepsilon, h_0^\varepsilon)$ satisfy the comptibility conditions of order r (see (2.4.15)) for some $r \ge 7$, and define h_1^ε by (2.4.20). In this section, we will prove that there is a unique vector field V solving the smoothed-out Euler equations (2.4.11) with h given by (2.4.7)-(2.4.8). We will work with the norms:

$$||V||_{\mathcal{X}^s(T)} = \sup{}_{0\le t\le T} ||V(t)||_{\mathcal{X}^s}, \quad (2.9.1)$$

where:

$$||V(t)||_{\mathcal{X}^s} = \textstyle\sum_{k=0}^{s-1} ||D_t^{1+k} V(t)||_{H^{s-k-1}(\Omega)} + ||V(t)||_{H^{s-1}(\Omega)}. \quad (2.9.2)$$

We let $\mathcal{X}^s(T)$ denote the closure of $C^\infty([0,T]; C^\infty(\overline{\Omega}))$ with respect to the norm $\mathcal{X}^s(T)$.

For a given vector field $V \in \mathcal{X}^{r+1}(T)$, we define the tangentially smoothed flow $\widetilde{x}$: $[0,T]\times\Omega \to \mathbb{R}^3$ as in (2.4.1), A as in (2.4.2) and the derivatives $\widetilde{\partial}, \widetilde{\Delta}$ by (2.4.3) and (2.4.4). By Theorem 2.F.1, if $h_0^\varepsilon, h_1^\varepsilon$ are compatible to order r, there is a function $h = h[V]$ which solves the problem:

$$D_t^2 e(h) - \widetilde{\Delta} h = (\widetilde{\partial}_i S_\varepsilon V^j)(\widetilde{\partial}_j V^i) - \rho(h), \qquad \text{on } [0,T]\times\Omega, \tag{2.9.3}$$

$$h = 0 \qquad \text{on } [0,T]\times\partial\Omega, \tag{2.9.4}$$

$$h(0,y) = h_0^\varepsilon(y), \quad D_t h(0,y) = h_1^\varepsilon(y), \qquad \text{on } \Omega, \tag{2.9.5}$$

and the estimates in Proposition 2.8.1 hold for h. We also define $\rho = \rho[h] = \rho[V]$ as in Section 2.4.2 and then define $\phi = \phi[V]$ by (2.4.9). We then define a map Λ by:

$$\Lambda^i(V)(t,y) = V_0^i(y) - \int_0^t \delta^{ij}\widetilde{\partial}_j h(s,y)\,ds - \int_0^t \delta^{ij}\widetilde{\partial}_j \phi(s,y)\,ds. \tag{2.9.6}$$

If V is a regular fixed point of Λ then it satisfies (2.4.11) and the corresponding h satisfies (2.9.4)-(2.9.5). Set:

$$E_0^s = ||V(t)||_{\mathcal{X}^s}\big|_{t=0}, \quad \mathcal{W}_0^s = \Big(\frac{1}{2}\sum\nolimits_{k\le s}\int_\Omega \big(|h_{k+1}^\varepsilon(y)|^2 + |\widetilde{\partial} h_k^\varepsilon(y)|^2\big)\widetilde{\kappa}_0(y)\,dy\Big)^{1/2}, \tag{2.9.7}$$

with the h_s^ε defined by (2.4.19) and $\widetilde{\kappa}_0 = \det(\partial x_0/\partial y)$. In Lemma 2.9.1, we show that if V satisfies (2.4.22), then these quantities are well-defined and bounded by the initial data. We also write:

$$e_0^s = e_1^s + e_2^s + e_2^s, \quad \text{where} \quad e_1^s = ||x_0||_{H^s(\Omega)}, \quad e_2^s = ||V_0^\varepsilon||_{H^s(\Omega)}, \quad e_3^s = ||h_0^\varepsilon||_{H^s(\Omega)}. \tag{2.9.8}$$

We remark that x_0 and h_0 are not independent; we take x_0 so that $\det(\partial x_0/\partial y) = 1/\rho(h_0)$, and consequently $e_3^s \le C(e_3^s)e_1^{s+1}$ for a constant C depending on the equation of state. However, it is more natural to state our estimates in terms of x_0^ε rather than h_0^ε in many cases so we will keep the notation separate.

The main result of this section is then:

Theorem 2.9.1. *Let $r \ge 7$. If $E_0^{r+1} + \mathcal{W}_r^0 + e_0^r < \infty$ and $V_0^\varepsilon, h_0^\varepsilon$ satisfy the compatibility conditions*

(2.4.15) *to order* r*, then for sufficiently small* ε*, there is a positive and continous function* $T_\varepsilon = T_\varepsilon(E_0^{r+1}, \mathcal{W}_r^0, e_0^r, \varepsilon^{-1})$ *so that for any* $0 \leq T \leq T_\varepsilon$*, there is a unique* $V \in \mathcal{X}^{r+1}(T)$ *which satisfies the smoothed-out problem* (2.4.11)*. Moreover, there is a positive continuous function* $\mathcal{F}_r$ *so that:*

$$\sup_{0\leq t\leq T}\big(||V(t)||_{\mathcal{X}^{r+1}} + ||\widetilde{\partial} h(t)||_r\big) \leq \varepsilon^{-1}\mathcal{F}_r(E_0^r, \mathcal{W}_{r-1}^0, e_0^{r-1})(E_0^{r+1} + \mathcal{W}_r^0 + e_0^r) + 1, \quad (2.9.9)$$

$$\sup_{0\leq t\leq T}\mathcal{W}_r(t) \leq \mathcal{F}_r(E_0^r, \mathcal{W}_{r-1}^0 e_0^r)\mathcal{W}_r^0 + 1. \quad (2.9.10)$$

First, in Section 2.9.1, we show that $\Lambda(V)$ is admissible (recall the definition in (2.4.22)) whenever V is, and that under the hypotheses of Theorem 2.9.1, the quantities $E_0^{r+1}, \mathcal{W}_r^0$ are bounded. In Proposition 2.9.1, we use the estimates from Section 2.8 to show that $||\Lambda(V)||_{\mathcal{X}^{r+1}}$ can be bounded in terms of the initial data, ε, and $||V||_{\mathcal{X}^{r+1}}$. This fact is then used in Corollary 2.9.1 to show that Λ maps a certain Banach space $\mathcal{C}^{r+1} \subset \mathcal{X}^{r+1}$ to itself (see (2.9.41)). Finally, in Proposition 2.9.2, we prove that if $V_I, V_{II} \in C^{r+1}$, $||\Lambda(V_I) - \Lambda(V_{II})||_{\mathcal{X}^r}$ can be bounded in terms of $||V_I||_{\mathcal{X}^{r+1}}, ||V_{II}||_{\mathcal{X}^{r+1}}$ and $||V_I - V_{II}||_{\mathcal{X}^r}$.

2.9.1 The initial data

Given $(V_0, h_0) \in H^r$ that satisfy the compatibility conditions (2.2.25) to order $r-1$, let $(V_0^\varepsilon, h_0^\varepsilon) \in H^r$ be the data constructed in Appendix 2.E that satisfy (2.4.15) to order $r-1$. Define h_1^ε by:

$$e'(h_0^\varepsilon)h_1^\varepsilon = -\operatorname{div} V_0^\varepsilon, \quad \text{where} \quad \operatorname{div} V_0^\varepsilon = \widetilde{\partial}_i V_0^i. \quad (2.9.11)$$

If $V_0^\varepsilon, ..., V_r^\varepsilon$ are as in (2.4.18), we will only consider vector fields V which are admissible to order r, meaning:

$$D_t^k V\big|_{t=0} = V_k^\varepsilon, \qquad k = 0, ..., r. \quad (2.9.12)$$

Taking M_0, L_0 so that:

$$|\partial x_0| + |\partial x_0^{-1}| + \textstyle\sum_{k+|J|\leq 3}|\partial^J V_k| \leq M_0/2, \quad \text{and} \quad \textstyle\sum_{k+|J|\leq 3}|\partial^J \widetilde{\partial} h_k| + |h_k| \leq L_0/2, \tag{2.9.13}$$

we have the following bounds for sufficiently small ε:

$$|A(0,y)| + |A^{-1}(0,y)| + \textstyle\sum_{k+|J|\leq 3}|\partial^J V_k^\varepsilon| \leq M_0, \quad \text{and} \quad \textstyle\sum_{k+|J|\leq 3}|\partial^J \widetilde{\partial} h_k^\varepsilon| + |h_k^\varepsilon| \leq L_0. \tag{2.9.14}$$

That there are data so that the compatibility conditions (2.4.15) hold follows from Theorem 2.E.1. We have:

Lemma 2.9.1. *Suppose that $(V_0^\varepsilon, h_0^\varepsilon)$ satisfy the compatibility conditions for smoothed Euler (2.4.15) to order $r-1$. Then if V satisfies (2.9.12), $\Lambda(V)$ also satisfies (2.9.12). Moreover, $E_0^{s+1} + \mathcal{W}_0^s \leq C(M_0)\big(e_1^{r+1} + P(e_0^r)\big)$, $s \leq r$.*

Proof. To see that $\Lambda(V)$ satisfies (2.9.12), note that $\Lambda(V)\big|_{t=0} = V_0^\varepsilon$, and for $k \geq 1$, by the definition of V_k^ε, (2.4.18):

$$D_t^k V|_{t=0} = V_k^\varepsilon = \textstyle\sum_{\ell\leq k} \widetilde{S}_{i\ell}^{jk}(\widetilde{\partial}\widetilde{V}_0^\varepsilon, ..., \widetilde{\partial}\widetilde{V}_{k-\ell-1}^\varepsilon)\widetilde{\partial}_j H_\ell^\varepsilon = -D_t^{k-1}\widetilde{\partial} h\big|_{t=0} - D_t^{k-1}\widetilde{\partial}\phi\big|_{t=0}, \tag{2.9.15}$$

where the last equality follows from the identity (2.4.16), and the fact that by construction $D_t^k h|_{t=0} = h_k^\varepsilon$, $D_t^k \phi|_{t=0} = \phi_k^\varepsilon$. The right-hand side here is $D_t^k \Lambda(V)$ by definition and this proves the first point.

To prove the second point, we start by showing that for $0 \leq s \leq r$,

$$E_0^{s+1} + W_0^s \leq C(M_0)(e_0^{s+1} + P(L_0, e_0^r)), \tag{2.9.16}$$

where L_0 is as in (2.9.13). If V satisfies (2.9.12), then:

$$D_t V|_{t=0} = V_1^\varepsilon = -\partial h_0^\varepsilon - \widetilde{\partial}\phi|_{t=0}, \tag{2.9.17}$$

and so:

$$E_0^1 = \big(||D_t V(t,\cdot)||_{L^2} + ||V(t,\cdot)||_{L^2}\big)\big|_{t=0} \leq C(M_0)\big(||h_0^\varepsilon||_{H^1} + ||\rho(h_0^\varepsilon)||_{L^2} + ||V_0^\varepsilon||_{L^2}\big), \quad (2.9.18)$$

where we used Theorem 2.7.1 to control $||\phi(0,\cdot,)||_{H^1}$. That W_0^0 is bounded by the right-hand side of (2.9.16) is immediate, so (2.9.16) hold for $s=0$. Suppose now that it holds for $s\leq m-1$. We introduce the notation:

$$e_0^{m^*} = \textstyle\sum_{k=0}^m ||h_k^\varepsilon||_{H^{m-k}}. \quad (2.9.19)$$

By definition we have $||V_0^\varepsilon||_{H^m} \leq e_0^m$. Suppose we know that for some $k\geq 0$:

$$||V_k^\varepsilon||_{H^{m-k}} \leq C(M_0)\big(||H_{k-1}^\varepsilon||_{H^{m-k+1}} + P(e_0^{m^*}, e_0^{m-1}, L_0)\big), \quad (2.9.20)$$

where H_{k-1}^ε is defined in Section 2.4.3, then by definition for F_k, and Theorem 2.7.4, we have:

$$\begin{aligned}||V_{k+1}^\varepsilon||_{H^{m-k-1}} &\leq C(M_0)\big(||H_k^\varepsilon||_{H^{m-k}} + ||(\partial V_k^\varepsilon)\partial H_0^\varepsilon||_{H^{m-k}} + ||F_{k+1}||_{H^{m-k-1}}\big)\\ &\leq C(M_0)\big(||h_k^\varepsilon||_{H^{m-k}} + P(e_0^{m^*} e_0^{m-1}, L_0)\big). \quad (2.9.21)\end{aligned}$$

Therefore (2.9.16) follows from bounding $e_0^{m^*}$ by $||V_0^\varepsilon||_{H^m}$ and $||h_0^\varepsilon||_{H^m}$, and hence e_0^m. To prove this, we use the continuity equation $h_1^\varepsilon = e'(h_0^\varepsilon)^{-1}\mathrm{div}\, V_0^\varepsilon$ which yields the bound $||h_1^\varepsilon||_{H^{m-1}} \leq P(L_0, e_0^m)$. In addition, suppose we know that for some $k\geq 3$

$$\textstyle\sum_{\ell\leq k-1} ||h_\ell^\varepsilon||_{H^{m-\ell}} \leq P(L_0, e_0^m).$$

We want to show that $||h_k||_{H^{m-k}}$ can be controlled by the same bound. This follows from the wave equation,

$$h_k^\varepsilon = e'(h_0^\varepsilon)^{-1}\big(\Delta h_{k-2}^\varepsilon + (\partial S_\varepsilon V_{k-2})(\partial V_0) + G_{k-2}\big), \quad (2.9.22)$$

where G_{k-2} is given in Section 2.4.3. This implies:

$$||h_k^\varepsilon||_{H^{m-k}} \le P(L_0, e_0^m), \tag{2.9.23}$$

where the bound of $||G_{k-2}||_{H^{m-k}}$ follows from Sobolev lemma. □

2.9.2 Existence on a time interval of size $O(\varepsilon)$

We can now prove Theorem 2.9.1. We start with the following simple lemma, which will be used to control some low norms of $\widetilde{x}$ and V.

Lemma 2.9.2. *Fix $r \ge 5$ and $T_1 > 0$ and suppose that V satisfies (2.9.12) and that $||V||_{\mathcal{X}^r(T_1)} \le K$. If the initial data satisfies (2.9.14), then there is a positive, continuous function D_0 so that if T satisfies:*

$$TD_0(M_0, K, T_1) \le 1, \text{ and } \quad T \le T_1, \tag{2.9.24}$$

then:

$$\sup_{0\le t\le T} \left(||\partial\widetilde{x}(t,\cdot)/\partial y||_{L^\infty} + ||\partial y(t,\cdot)/\partial\widetilde{x}||_{L^\infty} + \textstyle\sum_{k+|J|\le 3}||D_t^k\partial_y^J V(t,\cdot)||_{L^\infty}\right) \le 4M_0. \tag{2.9.25}$$

Proof. We integrate in time and use Sobolev embedding to get:

$$||\partial\widetilde{x}(t,\cdot)/\partial y(t,\cdot)||_{L^\infty} + \textstyle\sum_{k+|J|\le 3}||\partial_y^J D_t V(t,\cdot)||_{L^\infty} \le M_0 + \displaystyle\int_0^t ||V(\tau)||_{\mathcal{X}^5}\, d\tau. \tag{2.9.26}$$

The right-hand side is bounded by $2M_0$ if $T \le \min(T_1, M_0/K)$. To control $\partial y/\partial\widetilde{x}$, let $N(t) = ||\partial y(t,\cdot)/\partial\widetilde{x}||_{L^\infty(\Omega)}$ and note that by (2.D.2) we have $dN/dt \le C_0 N^2||\partial S_\varepsilon V||_{L^\infty(\Omega)}$. Using $N(0) \le M_0$ and Sobolev embedding, this implies that $N(t) \le M_0(1 - C_0 M_0 t||\partial V(t)||_{L^\infty(\Omega)})^{-1}$ for a constant C_0 depending only on Ω. Taking $D_0 = 2\min(K/M_0, 2C_0M_0^2)$, this implies that $N(t) \le 2M_0$ provided $T \le T_1$ and $TD_0 \le 1$. □

We can now begin the proof of local well-posedness. We will use the next estimate to show that Λ maps a certain Banach space to itself. We set $\overline{E}_0^r = E_0^{r+1} + \mathcal{W}_r^0 + e_0^r$.

Proposition 2.9.1. *Fix $r \geq 7$ and suppose that V_0, h_0 are such that $\overline{E}_0^r < \infty$. There are continuous, positive functions D_r, D_r' and polynomials $\mathcal{P}_1, \mathcal{P}_2$ so that the following statement holds: If T, ε satisfy:*

$$T D_r(M_0, L_0, \overline{E}_0^r) \leq 1, \qquad \varepsilon D_r'(M_0, L_0, \overline{E}_0^r) \leq 1, \tag{2.9.27}$$

and $V \in \mathcal{X}^{r+1}(T)$ is any vector field satisfying the condition (2.9.12) with $||V||_{\mathcal{X}^{r+1}(T)} \leq \varepsilon^{-2} E_0^{r+1}$, then:

$$\sup_{0 \leq t \leq T} ||\Lambda(V)(t)||_{\mathcal{X}^{r+1}}$$

$$\leq \varepsilon^{-1} \mathcal{P}_1\big(E_0^{r+1}, e_0^r, \mathcal{W}_r^0\big) + \varepsilon^{-1} T \mathcal{P}_2\big(E_0^{r+1}, e_0^r, \mathcal{W}_r^0, ||V||_{\mathcal{X}^{r+1}(T)}\big). \tag{2.9.28}$$

Proof. To get started, we fix $T_1 \leq 1$ and ε small enough that (2.9.14) holds, and consider only V so that $\sup_{0 \leq t \leq T_1} ||V(t)||_{\mathcal{X}^{r+1}} \leq \varepsilon^{-2} E_0^{r+1}$. With $\mathcal{F}_1 = -(\widetilde{\partial}_i S_\varepsilon V^j)(\widetilde{\partial}_j V^i)$, we fix $K_0 = K_0(E_0^{r+1}, e_0^r, \varepsilon^{-1})$ so that:

$$\sup_{0 \leq t \leq T_1} \big(||\widetilde{x}(t)||_r + ||V(t)||_{\mathcal{X}^{r+1}} + ||\mathcal{F}_1(t)||_{r,0} + ||\mathcal{F}_1(t)||_{r-1}\big) \leq K_0, \tag{2.9.29}$$

whenever $||V||_{\mathcal{X}^{r+1}(T_1)} \leq \varepsilon^{-2} E_0^{r+1}$, where we are bounding $||\widetilde{x}(t)||_r \leq ||\widetilde{x}(0)||_r + T_1 ||V||_{\mathcal{X}^{r+1}(T_1)}$.

Let D_0 be as in Lemma 2.9.2, and take T smaller if needed so that $T D_0(M_0, T_1) \leq 1$. By Lemma 2.9.2:

$$\sup_{0 \leq t \leq T} \big(||\partial \widetilde{x}(t, \cdot)/\partial y||_{L^\infty} + ||\partial y(t, \cdot)/\partial \widetilde{x}||_{L^\infty} + \textstyle\sum_{k+|J| \leq 3} ||D_t^k \partial_y^J V(t, \cdot)||_{L^\infty}\big) \leq 4M_0, \tag{2.9.30}$$

for all V satisfying (2.9.12) with $||V||_{\mathcal{X}^r(T)} \leq \varepsilon^{-2} E_0^{r+1}$. In particular, the assumption (2.5.2) holds with $M = 4M_0$.

With Q_r as in Corollary 2.6.1 and G_r' as in Theorem 2.F.1, we take T smaller again if necessary so that:

$$T\big(Q_r(4M_0, L_0, \mathcal{W}_r^0, K_0, T_1) + G_r'(4M_0, L_0, \mathcal{W}_r^0, K_0, T_1)\big) \leq 1. \tag{2.9.31}$$

By Lemma 2.9.1 and Theorem 2.F.1 the wave equation (2.9.4)-(2.9.5) has a unique solution $h = h[V]$ on $[0, T] \times \Omega$. By the above calculations and the first bound in (2.9.31), applying Proposition 2.8.1, for $0 \leq t \leq T$ we have:

$$\begin{aligned}||h(t)||_{s+1,0} &+ ||\widetilde{\partial} h(t)||_{H^s(\Omega)} \\ &\leq (1+\varepsilon^{-1})\mathcal{D}_1(||J_\varepsilon x(t)||_{H^r(\Omega)} + ||V(t)||_{H^r(\Omega)}) + T\mathcal{D}_2||V||_{\mathcal{X}^{r+1}(T)}, \quad 0 \leq s \leq r.\end{aligned} \tag{2.9.32}$$

where $\mathcal{D}_1, \mathcal{D}_2$ depend on $M_0, L_0, \mathcal{W}_r^0$ as well as $\sup_{0\leq t\leq T} ||\widetilde{x}(t)||_{H^r(\Omega)} + ||V(t)||_{\mathcal{X}^r}$. We now bound $||V(t)||_{H^r(\Omega)} \leq E_0^{r+1} + T||V||_{\mathcal{X}^{r+1}(T)}$ and $||\widetilde{x}(t)||_{H^r(\Omega)} + ||V(t)||_{\mathcal{X}^r} \leq E_0^{r+1} + e_0^r + T||V||_{\mathcal{X}^{r+1}(T)}$. This gives:

$$\sum_{s\leq r-1} ||h(t)||_{s+1,0} + ||\widetilde{\partial} h(t)||_{H^s(\Omega)} \leq (1+\varepsilon^{-1})\mathcal{D}_1' + T\mathcal{D}_2', \tag{2.9.33}$$

with

$$\mathcal{D}_1' = \mathcal{D}_1'(M_0, L_0, E_0^{r+1}, e_0^r, \mathcal{W}_r^0, T||V||_{\mathcal{X}^{r+1}(T)}) \tag{2.9.34}$$

and

$$\mathcal{D}_2' = \mathcal{D}_2'(M_0, L_0, E_0^{r+1}, e_0^r, \mathcal{W}_r^0, ||V||_{\mathcal{X}^{r+1}(T)}). \tag{2.9.35}$$

By Theorem 2.7.4, we also have:

$$\begin{aligned}||\widetilde{\partial}\phi||_r &\leq (1+\varepsilon^{-1})P\big(||\widetilde{x}||_{H^r(\Omega)}, ||V||_{\mathcal{X}^r}, ||V||_{H^r(\Omega)}, ||h||_{r-1}\big)||J_\varepsilon x||_{H^r(\Omega)}||h||_r \\ &\leq (1+\varepsilon^{-1})\mathcal{D}_3'(E_0^r, e_0^r) + T\mathcal{D}_4'\big(E_0^r, e_0^r, ||V||_{\mathcal{X}^{r+1}(T)}\big).\end{aligned} \tag{2.9.36}$$

Set $\mathcal{P}_1 = \mathcal{D}_1' + \mathcal{D}_3'$ and $\mathcal{P}_2 = \mathcal{D}_2' + \mathcal{D}_4'$. Since, for $k \geq 1$ we have $D_t^k \Lambda(V)_i = -D_t^{k-1}\widetilde{\partial}_i h -$

$D_t^{k-1}\widetilde{\partial}_i\phi$, the estimates for $||D_t^k\Lambda(V)||_{H^\ell(\Omega)}$ follow from (2.9.33)-(2.9.36), and the estimate

$$||\Lambda(V)(t)||_{H^r(\Omega)} \leq ||V_0||_{H^r(\Omega)} + T\sup_{0\leq\tau\leq T}\big(||\widetilde{\partial}h(\tau)||_{H^r(\Omega)} + ||\widetilde{\partial}\phi(\tau)||_{H^r(\Omega)}\big). \qquad \square$$

Corollary 2.9.1. *If the hypotheses of the previous theorem hold, there are positive, continuous, functions D_r'', $\mathcal{P}_r = \mathcal{P}_r(M_0, \overline{E}_0^r, \varepsilon)$ so that if T satisfies:*

$$TD_r''(M_0, \overline{E}_0^r, \varepsilon) \leq 1, \tag{2.9.37}$$

and if $V \in \mathcal{X}^{r+1}(T)$ satisfies (2.9.12) as well as the bound:

$$\sup_{0\leq t\leq T}||V(t)||_{\mathcal{X}^{r+1}} \leq \varepsilon^{-1}\mathcal{P}_r + 1, \tag{2.9.38}$$

then $\Lambda(V)$ satisfies:

$$\sup_{0\leq t\leq T}||\Lambda(V)||_{\mathcal{X}^{r+1}} \leq \varepsilon^{-1}\mathcal{P}_r + 1. \tag{2.9.39}$$

Proof. Let $D_r, D_r', \mathcal{P}_1, \mathcal{P}_2$ be as in Proposition 2.9.1. Take ε, T small enough that (2.9.27) holds. Let $T^*, \mathcal{P}_1, \mathcal{P}_2$ be as in Proposition 2.9.1. By Sobolev embedding and the elliptic estimate (2.8.9), we have that $L_0 \leq C_0(M_0, \overline{E}_0^r)$, and we take $D_r'' = D_r'(M_0, C_0, \overline{E})$. Now set $\mathcal{P}_r = \mathcal{P}_1$. Taking ε smaller if needed, the right-hand side of (2.9.38) is smaller than $\varepsilon^{-2}E_0^{r+1}$, and so if V satisfies (2.9.38) for $T \leq T^*$, then Proposition 2.9.1 applies and so:

$$\sup_{0\leq t\leq T}||\Lambda(V)(t)||_{\mathcal{X}^{r+1}} \leq \varepsilon^{-1}\mathcal{P}_1 + T\big(\varepsilon^{-1}\mathcal{P}_2(E_0^{r+1}, e_0^r, \mathcal{W}_r^0, \varepsilon^{-1}\mathcal{P}_1 + 1)\big). \tag{2.9.40}$$

We now take T small enough that this last factor is 1, which gives the result. $\square$

We now take T small enough that (2.9.37) holds and define:

$$\mathcal{C}^{r+1}(T) = \{V : [0,T]\times\Omega \to \mathbb{R}^3 \big| V \text{ satisfies (2.9.12)}$$

$$\text{and } \sup_{0\le t\le T} ||V(t)||_{\mathcal{X}^{r+1}} \le \varepsilon^{-1}\mathcal{P}_r(e_0^r, E_0^{r+1}) + 1\}. \quad (2.9.41)$$

Corollary 2.9.1 and Lemma 2.9.1 imply that $\Lambda : \mathcal{C}^{r+1}(T) \to \mathcal{C}^{r+1}(T)$. We now want to show that Λ has a fixed point in $\mathcal{C}^{r+1}(T)$ for T taken small enough. We start with:

Proposition 2.9.2. *Fix $r \ge 7$. There is a polynomial $\mathcal{P}_3 = \mathcal{P}_3(\overline{E}_0^r, \varepsilon^{-1})$ so that if T satisfies (2.9.37) for any V_I, V_{II} in $\mathcal{C}^{r+1}(T)$:*

$$\sup_{0\le t\le T} ||\Lambda(V_I)(t) - \Lambda(V_{II})(t)||_{\mathcal{X}^r} \le \varepsilon^{-1} T \mathcal{P}_3 ||V_I - V_{II}||_{\mathcal{X}^r}, \quad (2.9.42)$$

Proof. First, note that by Corollary 2.8.1 and Corollary 2.9.1, under our hypotheses we have:

$$\sup_{0\le t\le T} (||h_J(t)||_{r+1,0} + ||\widetilde{\partial} h_J(t)||_r) \le \mathcal{C}_1(M_0, \overline{E}_0^r, \varepsilon^{-1}) + 1, \quad (2.9.43)$$

for $J = I, II$ and some positive continuous function $\mathcal{C}_1$. For $k \ge 1$, we have:

$$D_t^k(\Lambda(V_I) - \Lambda(V_{II})) = D_t^{k-1}(\widetilde{\partial}_I h_I - \widetilde{\partial}_{II} h_{II}) + D_t^{k-1}(\widetilde{\partial}_I \phi_I - \widetilde{\partial}_{II}\phi_{II}). \quad (2.9.44)$$

By (2.8.20) combined with (2.8.19), we have:

$$||D_t^{k-1}(\widetilde{\partial}_I h_I(t) - \widetilde{\partial}_{II} h_{II}(t))||_{H^{r-k}} \le \mathcal{C}_r'\Big(||V_I(t) - V_{II}(t)||_{\mathcal{X}^{r-1}} + ||\widetilde{x}_I(t) - \widetilde{x}_{II}(t)||_{H^{r-1}}$$

$$+ \int_0^t ||V_I(\tau) - V_{II}(\tau)||_{\mathcal{X}^r}\Big), \quad (2.9.45)$$

and by (2.7.25):

$$||D_t^{k-1}(\widetilde{\partial}_I\phi_I - \widetilde{\partial}_{II}\phi_{II})||_{H^{r-k}}$$

$$\leq C_r' P(||h_I||_{r-1}, ||h_{II}||_{r-1})(||V_I - V_{II}||_{\mathcal{X}^r} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r})||h_I - h_{II}||_{r-1}, \quad (2.9.46)$$

where $C_r' = C_r'\big(M_0, \overline{E}_0^r, \sup_{0\leq t\leq T}(||\widetilde{x}_I(t)||_{H^r(\Omega)} + ||\widetilde{x}_{II}(t)||_{H^r(\Omega)}), ||V_I||_{\mathcal{X}^{r+1}(T)}, ||V_{II}||_{\mathcal{X}^{r+1}(T)}\big)$ and where we have used that $|\rho_I - \rho_{II}| = |\rho(h_I) - \rho(h_{II})| \leq C|\rho'||h_I - h_{II}|$. Combining these with the simple estimate:

$$\sup_{0\leq t\leq T}||V_I(t) - V_{II}(t)||_{\mathcal{X}^r} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^{r-1}} \leq 2T \sup_{0\leq t\leq T}||V_I(t) - V_{II}(t)||_{\mathcal{X}^{r+1}}, \quad (2.9.47)$$

and using (2.9.43), we have (2.9.42). □

Proof of Theorem 2.9.1. With notation as in Corollary 2.9.1 and Proposition 2.9.2, take ε so $\varepsilon/\mathcal{P}_3 \leq 1/D_r''$ and set

$$T_\varepsilon = 2^{-1}\varepsilon/\mathcal{P}_3. \quad (2.9.48)$$

If $T \leq T_\varepsilon$, by Lemma 2.9.1 and Corollary 2.9.1, for any $V \in \mathcal{C}^{r+1}(T)$, we have that $\Lambda(V) \in \mathcal{C}^{r+1}(T)$. Moreover, by Proposition 2.9.2, for any $V_I, V_{II} \in \mathcal{C}^{r+1}(T)$ we have that:

$$||\Lambda(V_I) - \Lambda(V_{II})||_{\mathcal{X}^r(T)} \leq 2^{-1}||V_I - V_{II}||_{\mathcal{X}^r(T)}. \quad (2.9.49)$$

With V_k^ε defined by (2.4.18), define the following sequence:

$$V^{(0)}(t,y) = \sum_{k=0}^r V_k^\varepsilon(y)t^k/k!, \qquad V^{(N)}(t,y) = \Lambda(V^{(N-1)})(t,y), \quad N \geq 1. \quad (2.9.50)$$

Noting that $D_t^s V^{(0)}|_{t=0} = V_k^\varepsilon$, by Corollary 2.9.1 and Lemma 2.9.1, the sequence $V^{(N)}$ is well defined and $V^{(N)} \in \mathcal{C}^{r+1}(T)$ for all N. Let $d_0 = \sup_{0\leq t\leq T}||\Lambda(V^{(0)})(t) - V^{(0)}(t)||_{\mathcal{X}^r}$. The estimate (2.9.49) implies:

$$||\Lambda(V^{(N)}) - \Lambda(V^{(M)})||_{\mathcal{X}^r(T)} \leq 2^{1-\min(M,N)} d_0, \tag{2.9.51}$$

for $0 \leq T \leq T_\varepsilon$. In particular the sequence $V^{(N)}$ is a Cauchy sequence in $\mathcal{X}^r(T)$. Let $V \in \mathcal{X}^r$ denote the limit. Because the norms $||V^{(N)}||_{\mathcal{X}^{r+1}(T)}$ are uniformly bounded we conclude that $V \in \mathcal{C}^{r+1}(T)$ as well.

The estimate (2.9.10) now follows from the definition of $\mathcal{C}^{r+1}(T)$ and the bounds in Corollary 2.8.1. □

2.10 Energy estimates

In the previous section, we constructed a solution to the smoothed problem on a time interval of size $O(\varepsilon)$. In this section, we prove the basic energy estimates which control Sobolev norms of the velocity uniformly in ε. We will not apply these energy estimates to the solutions V constructed in the previous section directly but instead to a sequence V_N which converges to V in an appropriate norm, and so we write our energy estimate in terms of remainders which we expect to converge to zero. In section 2.11 we prove estimates for the differentiated problem which will be used to show that these remainders do converge to zero. Finally, in section 2.12 we implement this strategy and prove the desired estimates for V, h satisfying Euler's equations.

The below energy estimates are slightly cumbersome, because we need to control a fractional number of derivatives of the solution V, and since Ω does not admit a global coordinate system, we will need to apply fractional derivatives in each coordinate patch separately. This unfortunately obscures the idea behind the estimates so let us explain how the energy estimates work in a simple case. The gravitational potential will not enter into the energy estimates to highest order, so we will ignore it for the moment. Let T be a vector field which is tangential at the boundary. Using the formula $[T, \widetilde{\partial}_i] = -(\widetilde{\partial}_i T\widetilde{x}^k)\widetilde{\partial}_k$, and applying T to the smoothed-out Euler's equations and the continuity equation (2.4.12), we have:

$$D_t TV_i - \widetilde{\partial}_i\big((T\widetilde{x}^k)\widetilde{\partial}_k h - Th\big) = -T\widetilde{x}^k\widetilde{\partial}_i\widetilde{\partial}_k h, \quad D_t Te(h) + \operatorname{div} TV = -(\widetilde{\partial}_i T\widetilde{x}^\ell)\widetilde{\partial}_\ell V^i \tag{2.10.1}$$

Multiplying the first equation by $TV^i\widetilde{\kappa}$ and integrating over Ω gives:

$$\frac{1}{2}\frac{d}{dt}||TV(t)||^2_{L^2(\Omega)} - \int_\Omega \widetilde{\partial}_i\big((T\widetilde{x}^k)\widetilde{\partial}_k h - Th\big)TV^i\,\widetilde{\kappa}dy = \text{ lower-order terms.} \tag{2.10.2}$$

Integrating by parts and using that $Th = 0$ on $\partial\Omega$, we have:

$$\begin{aligned} &-\int_\Omega \widetilde{\partial}_i\big((T\widetilde{x}^k)\widetilde{\partial}_k h - Th\big)TV^i\,\widetilde{\kappa}dy \\ &\qquad = -\int_{\partial\Omega}(T\widetilde{x}^k)(TV^i)N_i\widetilde{\partial}_k h\widetilde{\kappa}dy + \int_\Omega((T\widetilde{x}^k)\widetilde{\partial}_k h - Th)\widetilde{\partial}_i TV^i\,\widetilde{\kappa}dy. \end{aligned} \tag{2.10.3}$$

We now manipulate the boundary term. Recall that $\widetilde{x} = S_\varepsilon x = J_\varepsilon J_\varepsilon x$ and that J_ε is symmetric with respect to the surface measure dS. Ignoring the commutator $[T, J_\varepsilon]$ for the moment, the boundary term is:

$$\begin{aligned} -\int_{\partial\Omega} J_\varepsilon(TJ_\varepsilon x^k)(TV^i)\widetilde{\partial}_k hN_i\,\widetilde{\nu}dS(y) &= -\int_{\partial\Omega}(TJ_\varepsilon x^k)(TJ_\varepsilon V^i)\widetilde{\partial}_k hN_i\,\widetilde{\nu}dS(y) \\ &\quad - \int_{\partial\Omega}(TJ_\varepsilon x^k)\big[J_\varepsilon\big(TV^i\widetilde{\partial}_k hN_i\widetilde{\nu}\big) - (J_\varepsilon TV^i)(\widetilde{\partial}_k hN_i\widetilde{\nu})\big]\,dS(y). \end{aligned} \tag{2.10.4}$$

Because $h = 0$ and $N \cdot \widetilde{\partial}h < 0$ on $\partial\Omega$, it follows that $\widetilde{\partial}_k h = -N_k|\widetilde{\partial}h|^{-1}$, so the first term here is the time derivative of a positive term to highest order:

$$-\int_{\partial\Omega}(TJ_\varepsilon x^k)(TJ_\varepsilon V^i)\widetilde{\partial}_k h N_i \widetilde{\nu} dS = \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega}(TJ_\varepsilon x^k)(TJ_\varepsilon x^i)N_k N_i|\widetilde{\partial} h|\widetilde{\nu} dS$$

$$+\int_{\partial\Omega}(TJ_\varepsilon x^k)(TJ_\varepsilon x^i)D_t(N_k N_i|\widetilde{\partial} h|\widetilde{\nu})dS. \quad (2.10.5)$$

Since J_ε is a convolution with a function supported on a ball of size $\sim \varepsilon$, one should expect that the second term in (2.10.4) is bounded by $C\varepsilon||T\widetilde{x}||_{L^2(\partial\Omega)}||TV||_{L^2(\partial\Omega)}$, with the constant depending on bounds for $\widetilde{\partial} h$.

Using the second equation in (2.10.1), the interior term is:

$$\int_\Omega ((T\widetilde{x}^k)\widetilde{\partial}_k h - Th)\widetilde{\partial}_i TV^i\,\widetilde{\kappa} dy$$

$$= \int_\Omega ((T\widetilde{x}^k)\widetilde{\partial}_k h - Th)TD_t e(h)\,\widetilde{\kappa} dy + \int_\Omega ((T\widetilde{x}^k)\widetilde{\partial}_k h - Th)(\widetilde{\partial}_i T\widetilde{x}^\ell)(\widetilde{\partial}_\ell V^i)\widetilde{\kappa} dy. \quad (2.10.6)$$

This leads to an energy identity of the form:

$$\frac{d}{dt}(||TV(t)||^2_{L^2(\Omega)} + ||TJ_\varepsilon x(t)\cdot N||^2_{L^2(\partial\Omega)} + ||\sqrt{e'(h)}Th||^2_{L^2(\Omega)})$$

$$\lesssim \left|\int_\Omega ((T\widetilde{x}^k)\widetilde{\partial}_k h - Th)(\widetilde{\partial}_i T\widetilde{x}^\ell)\widetilde{\partial}_\ell V^i \widetilde{\kappa} dy\right|$$

$$+\varepsilon||T\widetilde{x}(t)||_{L^2(\partial\Omega)}||TV(t)||_{L^2(\partial\Omega)} + \text{ lower order terms.} \quad (2.10.7)$$

By the elliptic estimates (2.5.10) the second term on the left controls $||TJ_\varepsilon x||_{L^2(\partial\Omega)}$ provided we have estimates for $\operatorname{div} TJ_\varepsilon x, \operatorname{curl} TJ_\varepsilon x$, and so one can think of this as controlling $||J_\varepsilon x||_{H^{3/2}(\Omega)}$ and thus $||\widetilde{x}||_{H^{3/2}(\Omega)}$, by the trace inequality. The term on the right-hand side looks problematic because we do not control $||\widetilde{x}||_{H^2(\Omega)}$, however we will be able to "integrate half a derivative by parts" in this term using (2.A.7) to control it.

To control the term $\varepsilon||TV(t)||_{L^2(\partial\Omega)}$, it turns out that it can be bounded by $||\gamma \cdot TV||_{L^2(\partial\Omega)}$ provided we control the divergence and curl appropriately in the interior, where γ is the projection to the tangent space at the boundary. We will see that $\varepsilon d/dt||\gamma \cdot TV||_{L^2(\partial\Omega)}$ is lower-order, uniformly in ε, because to highest order $D_t\gamma \cdot TV \sim \gamma \cdot T\widetilde{\partial} h \sim \gamma \cdot \widetilde{\partial} Th + T\partial\widetilde{x} \cdot \partial h$. The first term is zero because $h = 0$ on $\partial\Omega$ and using the smoothing property (2.A.23) the second term is $O(\varepsilon^{-1})$.

2.10.1 Higher order energy estimates

Let α be a vector field on Ω and q a function with $q = 0$ on $\partial\Omega$. We suppose that the following hold:

$$|D_t\widetilde{\partial} q|/|\widetilde{\partial} q| + |q| \leq K, \qquad \text{on } \partial\Omega, \tag{2.10.8}$$

$$|D_t^k \partial_y^I q| \leq K, \quad k + |I| \leq 3, \quad \text{in } \Omega. \tag{2.10.9}$$

As in earlier sections, we will also fix a strictly positive function σ so that $|\sigma'| \lesssim \sigma$. We also let $x \in C^1([0, T]; H^r(\Omega))$ for $r \geq 7$ be a given vector field and let $V = D_t x$.

With notation as in (2.3.17) and $\langle\partial_\theta\rangle_\mu^{1/2}$ defined by (2.3.10), for each $T^I \in \mathcal{T}^s$, we define $E_\mu^I = E_{\mu,1}^I + E_{\mu,2}^I$, where:

$$E_{\mu,1}^I = \frac{1}{2}\int_\Omega \delta^{ij}(T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha_i)(T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha_j)\widetilde{\kappa} dy + \frac{1}{2}\int_\Omega \sigma|T^I\langle\partial_\theta\rangle_\mu^{1/2} q|^2 \widetilde{\kappa} dy, \tag{2.10.10}$$

$$E_{\mu,2}^I = \frac{1}{2}\int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2} J_\varepsilon x^i)(T^I\langle\partial_\theta\rangle_\mu^{1/2} J_\varepsilon x^j) N_i N_j |\widetilde{\partial} q|\, \widetilde{\nu} dS, \tag{2.10.11}$$

as well as:

$$E^I = \textstyle\sum_{\mu=0}^N E_{\mu,1}^I + E_{\mu,2}^I, \qquad E^s = \sum_{|I|\leq s} E^I. \tag{2.10.12}$$

Here $\widetilde{\kappa} dy = d\widetilde{x}$ is the volume form on $\widetilde{\mathcal{D}}_t$ and $\widetilde{\nu}$ is such that $\widetilde{\nu}$ times the surface measure on $\partial\Omega$ is equal to the surface measure on $\partial\widetilde{\mathcal{D}}_t$ in the $\widetilde{x}$ coordinates. We will also need to control the time derivative of

$$E^I_{\mu,\varepsilon} = \int_{\partial\Omega} \gamma^{ij} (\langle\partial_\theta\rangle_\mu^{1/2} T^I \alpha_i)(\langle\partial_\theta\rangle_\mu^{1/2} T^I \alpha_j)\, \widetilde{\nu} dS, \quad E^I_\varepsilon = \sum_{\mu=K}^{N} E^I_{\mu,\varepsilon}, \quad E^s_\varepsilon = \sum_{|I|\leq s} E^I_\mu. \tag{2.10.13}$$

The fact that the fractional derivative operator $\langle\partial_\theta\rangle_\mu^{1/2}$ appears on the "outside" in this definition and the "inside" in the definition of the E^I_μ is just to make the computation simpler and has no special significance; note that the commutator $[T^I, \langle\partial_\theta\rangle_\mu^{1/2}]$ is an operator of lower order by Lemma 2.A.1. The quantity E^s_ε will appear in our calculation weighted with a power of ε and will be needed in order to show that we have a solution to the problem (2.4.11) on a time interval independent of ε.

We write $T^I = ST^J$ where $S \in \mathcal{T}$ and $|J| = s-1$ and define:

$$R^I = \sum_{\mu=1}^N ||D_t T^I \langle\partial_\theta\rangle_\mu^{1/2} \alpha - \widetilde{\partial}\big((T^I \langle\partial_\theta\rangle_\mu^{1/2} \widetilde{x}^j)(\widetilde{\partial}_j q) + T^I \langle\partial_\theta\rangle_\mu^{1/2} q\big)||_{L^2(\Omega)}$$

$$+ ||\langle\partial_\theta\rangle_\mu^{1/2}\big(\sigma D_t T^J q - (\widetilde{\partial}_i T^J \widetilde{x}^j)\widetilde{\partial}_j V^i + \mathrm{div}\, T^J \alpha\big)||_{L^2(\Omega)} + ||D_t T^I \langle\partial_\theta\rangle_\mu^{1/2} x - T^I \langle\partial_\theta\rangle_\mu^{1/2} \alpha||_{L^2(\partial\Omega)}, \tag{2.10.14}$$

and $R^s = \sum_{|I|\leq s} R^I$. We will ultimately take $\alpha = V$ and $q = h$, in which case using (2.10.1), one expects the first two terms here to be lower order. See Section 2.11. We also define:

$$R^I_\varepsilon = \sum_{\mu=1}^N ||\gamma \cdot \langle\partial_\theta\rangle_\mu^{1/2} T^I D_t \alpha - (\langle\partial_\theta\rangle_\mu^{1/2} T^I \gamma) \cdot \widetilde{\partial} q||_{L^2(\partial\Omega)}, \quad R^s_\varepsilon = \sum_{|I|\leq s} R^I_\varepsilon. \tag{2.10.15}$$

The main result of this section is:

Theorem 2.10.1. *Suppose that $\widetilde{x}$ satisfies the assumption (2.5.2), that α and q are given as above and that the assumptions (2.10.8)-(2.10.9) hold. With $E^I, E^s, E^I_\varepsilon, E^s_\varepsilon$ defined as in (2.10.12)-(2.10.13) and*

R^s, R^s_ε defined by (2.10.14)-(2.10.15)*, there is a continuous function $C = C(M, K)$ so that:*

$$\frac{d}{dt}E^I \leq C\sqrt{E^s}\Big(R^s + ||\mathcal{T}^{s-1}\alpha||_{H^{(1,1/2)}(\Omega)} + ||\mathcal{T}^{s-1}J_\varepsilon x||_{H^1(\Omega)}$$

$$+ ||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)} + ||\mathcal{T}^s q||_{H^1(\Omega)} + ||D_t\mathcal{T}^s q||_{L^2(\Omega)}\Big)$$

$$+ CE^s + C\varepsilon||\alpha||_{H^{s+1/2}(\partial\Omega)}||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}, \qquad |I| \leq s, \quad (2.10.16)$$

and

$$\frac{d}{dt}E^I_\varepsilon \leq C\sqrt{E^s_\varepsilon}\big(R^s_\varepsilon + \varepsilon^{-1}||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}\big). \qquad (2.10.17)$$

In particular, writing $E = E^s + \varepsilon^2 E^s_\varepsilon$ and $R = R^s + \varepsilon^2 R^s_\varepsilon$, we have:

$$\frac{d}{dt}E \leq C\sqrt{E}\Big(R + ||\mathcal{T}^{s-1}\alpha||_{H^{(1,1/2)}(\Omega)} + ||\mathcal{T}^{s-1}J_\varepsilon x||_{H^1(\Omega)} + ||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}$$

$$+ ||\mathcal{T}^s q||_{H^1(\Omega)} + ||D_t\mathcal{T}^s q||_{L^2(\Omega)}\Big)$$

$$+ C_s E + \varepsilon||\alpha||_{H^{s+1/2}(\partial\Omega)}||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}. \qquad (2.10.18)$$

These estimates appear to lose half a derivative since E^s only controls $||\mathcal{T}^s q||_{H^{(0,1/2)}(\Omega)}$ and not $||\mathcal{T}^s q||_{H^1(\Omega)}$, but q will satisfy a wave equation which gains enough regularity to close these estimates; see Lemma 2.12.1.

Proof. We will prove that $d(E^I_{\mu,1} + E^I_{\mu,2})/dt$ is bounded by the right-hand side of (2.10.16). We

have:

$$\frac{d}{dt}E^I_{\mu,1} = \int_\Omega \delta^{ij}\Big(T^I\langle\partial_\theta\rangle^{1/2}_\mu D_t\alpha_i - \widetilde{\partial}_i\big((T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k)(\widetilde{\partial}_k q) + T^I\langle\partial_\theta\rangle^{1/2}_\mu q\big)\Big)(T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha_j)\,\widetilde{\kappa} dy$$

$$+\int_\Omega \sigma(D_t T^I\langle\partial_\theta\rangle^{1/2}_\mu q)(T^I\langle\partial_\theta\rangle^{1/2}_\mu q)\widetilde{\kappa} dy + \int_\Omega \delta^{ij}\widetilde{\partial}_i\Big((T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k)(\widetilde{\partial}_k q) - T^I\langle\partial_\theta\rangle^{1/2}_\mu q\Big)(T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha_j)\,\widetilde{\kappa} dy$$

$$+\int_\Omega \Big(\delta^{ij}(T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha_i)(T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha_j) + \sigma|T^I\langle\partial_\theta\rangle^{1/2}_\mu q|^2\Big)(D_t\widetilde{\kappa}) + (D_t\sigma)|T^I\langle\partial_\theta\rangle^{1/2}_\mu q|^2\,\widetilde{\kappa} dy. \tag{2.10.19}$$

The first and the last terms and are bounded by (2.10.16). After integrating by parts, using Green's formula (2.A.50), and the facts that $T^I\langle\partial_\theta\rangle^{1/2}_\mu q = 0$ and $\widetilde{\partial}_k q = -N_k|\widetilde{\partial} q|$ on $\partial\Omega$, the second term on the second line is:

$$-\int_{\partial\Omega}(T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k)(T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha_j)N_j N_k|\widetilde{\partial} q|\,\widetilde{\nu} dS$$

$$-\int_\Omega (T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k(\widetilde{\partial}_k q) - \langle\partial_\theta\rangle^{1/2}_\mu T^I q)(\mathrm{div}\, T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha)\,\widetilde{\kappa} dy. \tag{2.10.20}$$

In order to handle the interior term, we need to perform a few manipulations. We start by writing $\mathrm{div}\, T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha = \big(\sigma D_t T^I\langle\partial_\theta\rangle^{1/2}_\mu q - (\widetilde{\partial}_i T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^j)\widetilde{\partial}_j V^i + \mathrm{div}\, T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha\big) - \sigma D_t T^I\langle\partial_\theta\rangle^{1/2}_\mu q + (\widetilde{\partial}_i T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^j)\widetilde{\partial}_j V^i$. Inserting this into (2.10.20), we get:

$$\int_\Omega \big(T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k(\widetilde{\partial}_k q) + T^I\langle\partial_\theta\rangle^{1/2}_\mu q\big)\big(\sigma D_t T^I\langle\partial_\theta\rangle^{1/2}_\mu q - (\widetilde{\partial}_i T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^j)\widetilde{\partial}_j V^i + \mathrm{div}\, T^I\langle\partial_\theta\rangle^{1/2}_\mu \alpha\big)\,\widetilde{\kappa} dy$$

$$+\int_\Omega \big(T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k(\widetilde{\partial}_k q)\big)(\sigma D_t T^I\langle\partial_\theta\rangle^{1/2}_\mu q)\,\widetilde{\kappa} dy + \int_\Omega \big(T^I\langle\partial_\theta\rangle^{1/2}_\mu q\big)(\widetilde{\partial}_i T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k \widetilde{\partial}_k V^i)\,\widetilde{\kappa} dy$$

$$+\int_\Omega \big(T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^k\big)\big(\widetilde{\partial}_i T^I\langle\partial_\theta\rangle^{1/2}_\mu \widetilde{x}^j\big)\widetilde{\partial}_j V^i\,\widetilde{\kappa} dy. \tag{2.10.21}$$

We now want to integrate half a derivative by parts so we write $T^I = ST^J$, $|J| = s-1$. Since

$\widetilde{\partial}_i = A^a_{\ i}\partial_a$, we have:

$$||[S, \text{div}]T^J\langle\partial_\theta\rangle_\mu^{1/2}\alpha||_{L^2(\Omega)} \leq C(M)||\partial T^J\langle\partial_\theta\rangle_\mu^{1/2}\alpha||_{L^2(\Omega)}, \tag{2.10.22}$$

$$||[S, \widetilde{\partial}]T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{L^2(\Omega)} \leq C(M)||\partial T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{L^2(\Omega)}. \tag{2.10.23}$$

These terms are bounded by (2.10.16). Writing $F_1 = \sigma D_t T^J\langle\partial_\theta\rangle_\mu^{1/2}q - (\widetilde{\partial}_i T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j)\widetilde{\partial}_j V^i + \text{div}\, T^J\langle\partial_\theta\rangle_\mu^{1/2}\alpha$, applying (2.A.7) and the Leibniz rule (2.A.8), we have:

$$\Big|\int_\Omega (-T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^k(\widetilde{\partial}_k q) + \sigma T^I\langle\partial_\theta\rangle_\mu^{1/2}q) S(F_1)\,\widetilde{\kappa} dy\Big| \leq CK(||T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{1/2} + ||T^I\langle\partial_\theta\rangle_\mu^{1/2}q||_{1/2})\times$$

$$||\sqrt{\sigma}D_t T^J\langle\partial_\theta\rangle_\mu^{1/2}q - (\widetilde{\partial}_i T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j)\widetilde{\partial}_j V^i + \text{div}\, T^J\langle\partial_\theta\rangle_\mu^{1/2}\alpha||_{1/2}, \tag{2.10.24}$$

where $||\cdot||_{1/2} = ||\cdot||_{H^{(0,1/2)}(\Omega)}$, as well as:

$$\Big|\int_\Omega (T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^k)(\widetilde{\partial}_k q) S(\sigma D_t T^J\langle\partial_\theta\rangle_\mu^{1/2}q)\widetilde{\kappa} dy\Big| + \Big|\int_\Omega (T^I\langle\partial_\theta\rangle_\mu^{1/2}q) S(\widetilde{\partial}_i T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j(\widetilde{\partial}_j V^i)\,\widetilde{\kappa} dy\Big|$$

$$\leq CK(||\sigma D_t T^J\langle\partial_\theta\rangle_\mu^{1/2}q||_{1/2} + ||\sigma T^I\langle\partial_\theta\rangle_\mu^{1/2}q||_{1/2})(||T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{1/2} + ||\widetilde{\partial}T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{1/2}), \tag{2.10.25}$$

and finally:

$$\Big|\int_\Omega (T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^k)(\widetilde{\partial}_k q) S(\widetilde{\partial}_i T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j(\widetilde{\partial}_j V^i))\,\widetilde{\kappa} dy\Big| \leq C||T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{1/2}||\widetilde{\partial}T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}||_{1/2}. \tag{2.10.26}$$

Since $||\langle\partial_\theta\rangle_\mu^{1/2}f||_{1/2} \lesssim ||f||_{H^1(\Omega)}$ using Lemma 2.A.1, the terms with $\widetilde{x}$ are bounded by (2.10.16).

It remains to control the boundary term in (2.10.20). Recalling the definition of $E_2^{I,\mu}$ from

(2.10.13), we have:

$$\frac{d}{dt}E^I_{\mu,2} - \int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^k)(T\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j)N_jN_k|\widetilde{\partial}q|\,\widetilde{\nu}dS$$

$$= \int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^k)(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon D_t x^j) - (T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^k)(T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j)N_jN_k|\widetilde{\partial}q|\,\widetilde{\nu}dS$$

$$+\frac{1}{2}\int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^k)(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^j)D_t(N_jN_k|\widetilde{\partial}q|\,\widetilde{\nu}dS). \quad (2.10.27)$$

The last term is bounded by (2.10.16) by the assumption (2.10.8). To handle the second term, we recall that $\widetilde{x} = S_\varepsilon x$, $S_\varepsilon = J_\varepsilon^2$ and that J_ε is symmetric with respect to the measure dS, so:

$$\int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^k)(T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j)N_jN_k|\widetilde{\partial}q|\,\widetilde{\nu}dS = \int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^k)(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon\alpha^j)N_jN_k|\widetilde{\partial}q|\,\widetilde{\nu}dS$$

$$+\int_{\partial\Omega}\Big(([T^I\langle\partial_\theta\rangle_\mu^{1/2},J_\varepsilon]J_\varepsilon x^k)(T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j) + (T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^k)([J_\varepsilon,T^I\langle\partial_\theta\rangle_\mu^{1/2}]\alpha^j)\Big)N_jN_k|\widetilde{\partial}q|\,dS$$

$$+\int_{\partial\Omega}(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^k)\Big(\big(J_\varepsilon(T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha^jN_jN_k|\widetilde{\partial}q|\big) - (J_\varepsilon T^I\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j)N_jN_k|\widetilde{\partial}q|)\Big)\,dS \quad (2.10.28)$$

The first term cancels the first term from (2.10.27). Integrating by parts in the first term on the second line and using (2.A.32) and (2.A.29), the terms on the second and third line are bounded by the right side of (2.10.16).

Finally, we control the time derivative of $E^I_{\mu,\varepsilon}$. We have:

$$\frac{d}{dt}E^I_{\mu,\varepsilon} = \int_{\partial\Omega}(\langle\partial_\theta\rangle_\mu^{1/2}T^I\alpha_i)(\langle\partial_\theta\rangle_\mu^{1/2}T^ID_t\alpha_j)\gamma^{ij}\,\widetilde{\nu}dS + \frac{1}{2}\int_{\partial\Omega}(\langle\partial_\theta\rangle_\mu^{1/2}T^I\alpha_i)(\langle\partial_\theta\rangle_\mu^{1/2}T^I\alpha_j)D_t(\gamma^{ij}\widetilde{\nu})dS. \quad (2.10.29)$$

The second term is bounded by (2.10.17). The idea is that $\gamma^{ij}\langle\partial_\theta\rangle_\mu^{1/2}T^I\widetilde{\partial}_iq$ is lower order because $q=0$ on $\partial\Omega$, the operators $T^I, \langle\partial_\theta\rangle_\mu^{1/2}$ are tangential, and we are multiplying by the tangential projection γ. We have:

$$||\langle\partial_\theta\rangle_\mu^{1/2}\gamma^{ij}||_{H^k(\partial\Omega)} \leq C(M)||\partial_y\widetilde{x}||_{H^{k+1/2}(\partial\Omega)}, \quad ||\gamma^{ij}||_{H^k(\partial\Omega)} \leq C(M)||\partial_y\widetilde{x}||_{H^k(\partial\Omega)}, \quad (2.10.30)$$

which follows from $\gamma^{ij} = \gamma^{ab}A^i_{\ a}A^j_{\ b}$, the fractional product rule (2.A.8), the formula (2.D.2), and interpolation.

We write $\gamma^{ij}\langle\partial_\theta\rangle_\mu^{1/2}T^I\widetilde{\partial}_i q = \langle\partial_\theta\rangle_\mu^{1/2}(\gamma^{ij}T^I\widetilde{\partial}_i q) + [\gamma^{ij}, \langle\partial_\theta\rangle_\mu^{1/2}]T^I\widetilde{\partial}_i q$. The $L^2(\partial\Omega)$ norm of the second term here is bounded by $C(M)||\widetilde{x}||_{H^3(\partial\Omega)}||T^I\widetilde{\partial}q||_{L^2(\partial\Omega)}$ by (2.10.30) and the fractional product rule (2.A.8). We then write $\gamma^{ij}T^I\widetilde{\partial}_i q = T^I(\gamma^{ij}\widetilde{\partial}_i q) - (T^I\gamma^{ij})\widetilde{\partial}_i q + \sum_{J+K=I, |J|,|K|\leq|I|-1}(T^J\gamma^{ij})(T^K\widetilde{\partial}_i q)$. Applying $\langle\partial_\theta\rangle_\mu^{1/2}$ to these terms, we see that the first term is zero because $q = 0$ on $\partial\Omega$, while:

$$\begin{aligned}||\langle\partial_\theta\rangle_\mu^{1/2}\big((T^I\gamma^{ij})\widetilde{\partial}_i q\big)||_{L^2(\partial\Omega)} &\leq ||\langle\partial_\theta\rangle_\mu^{1/2}T^I\gamma^{ij}||_{L^2(\partial\Omega)}||\widetilde{\partial}q||_{H^3(\partial\Omega)} \\ &\leq C(M)||\langle\partial_\theta\rangle_\mu^{1/2}T^I\partial_y\widetilde{x}||_{L^2(\partial\Omega)}||\widetilde{\partial}q||_{H^3(\partial\Omega)}. \quad (2.10.31)\end{aligned}$$

Since $|I| = s$ this last term is higher-order than (2.10.17), so we write $T^I = ST^J$, $S \in \mathcal{T}$ and use the smoothing property $||SJ_\varepsilon f||_{L^2(\partial\Omega)} \lesssim \varepsilon^{-1}||f||_{L^2(\partial\Omega)}$ to bound it by $\varepsilon^{-1}||\langle\partial_\theta\rangle_\mu^{1/2}T^J\partial_y J_\varepsilon x||_{L^2(\partial\Omega)} \lesssim \varepsilon^{-1}||\partial_y J_\varepsilon x||_{H^{s-1/2}(\partial\Omega)}$.

It now remains to control $||\langle\partial_\theta\rangle_\mu^{1/2}\big((T^J\gamma^{ij})(T^K\widetilde{\partial}_i q)\big)||_{L^2(\partial\Omega)}$ and for this we use the Leibniz rule (2.A.8) in a few different ways. First, when $s \leq 5$ we bound the result by $||T^J\gamma||_{H^2(\partial\Omega)}||\langle\partial_\theta\rangle_\mu^{1/2}T^K\widetilde{\partial}q||_{L^2(\partial\Omega)}$ and this is bounded by the right-hand side of (2.10.17). If $s \geq 6$ and $|K| \leq s-3$ we bound this by $||\langle\partial_\theta\rangle_\mu^{1/2}T^J\gamma^{ij}||_{L^2(\partial\Omega)}||T^K\widetilde{\partial}q||_{H^2(\partial\Omega)}$ and if $|K| \geq s-2$ then since $s \geq 6$ and $|J|, |K| \leq s-1$, we have $|J| \leq s-3$ and so we bound it by $||T^J\gamma^{ij}||_{H^2(\partial\Omega)}||\langle\partial_\theta\rangle_\mu^{1/2}T^K\widetilde{\partial}q||_{L^2(\partial\Omega)}$. In each of these cases, applying (2.10.30) we wind up with terms which are bounded by the right-hand side of (2.10.17). This completes the proof. □

2.11 The higher-order equations

Fix $r \geq 5$ and let $V \in \mathcal{X}^{r+1}(T)$ be the solution to the smoothed-out Euler equations (2.4.11) constructed in Section 2.9. Recall that if h is the corresponding enthalpy, then we have:

$$V \in L^\infty(0,T;H^r(\Omega)), \qquad D_t^k V \in L^\infty(0,T;H^{r+1-k}(\Omega)), \qquad k = 1, ..., r+1, \tag{2.11.1}$$

$$D_t^{r+1} h \in L^\infty(0,T;L^2(\Omega)), \qquad D_t^k \widetilde{\partial} h \in L^\infty(0,T;H^{r-k}(\Omega)), \qquad k = 0, ..., r. \tag{2.11.2}$$

In this section it is convenient to assume that we have a bit more regularity of $\widetilde{x}$. We will assume that:

$$|A^i{}_a| + |A^a{}_i| + ||\widetilde{x}||_{H^6(\Omega)} \leq M', \quad \text{ on } \Omega. \tag{2.11.3}$$

The reason we want this assumption is that the fractional product rule (2.A.8) involves Sobolev norms. We will use notation similar but not identical to that in Sections 2.5 and 2.7 and let $C_0, C_s, s \geq 1, C_s'$ denote a continuous function of the following arguments:

$$C_0 = C_0(M'), \qquad C_s = C_s(M', ||\widetilde{x}||_{H^s(\Omega)}), \qquad C_s' = C_s'(M', ||\widetilde{x}||_{H^s(\partial\Omega)}). \tag{2.11.4}$$

In order to prove that we have uniform energy estimates for V, we need to show that we can control R^s and R^s_ε in terms of the energy. The first step is the following:

Lemma 2.11.1. *Let $r \geq 5$ and let $V \in \mathcal{X}^{r+1}(T)$ be the solution to the smoothed-out Euler equations constructed in the previous section. Suppose that (2.11.3) holds. Fix $1 \leq s \leq r-1$, let $T^I \in \mathcal{T}^s$ and write $T^I = ST^J$ for $S \in \mathcal{T}, |J| = s-1$. For each $\mu, \nu = 0, ..., N$, we have:*

$$||D_t T^I \langle \partial_\theta \rangle_\mu^{1/2} V - \widetilde{\partial}\big((T^I \langle \partial_\theta \rangle_\mu^{1/2} \widetilde{x}^j)(\widetilde{\partial}_j h) + T^I \langle \partial_\theta \rangle_\mu^{1/2} h\big)||_{L^2(\Omega)}$$

$$\leq C_s ||\widetilde{x}||_{H^s(\Omega)} (||\widetilde{\partial} h||_{H^s(\Omega)} + ||\widetilde{\partial} h||_{H^2(\Omega)}) + ||\langle \partial_\theta \rangle_\mu^{1/2} T^I \widetilde{\partial} \phi(h)||_{L^2(\Omega)}, \tag{2.11.5}$$

$$||\langle\partial_\theta\rangle_\nu^{1/2}\big(e'(h)D_tT^J\langle\partial_\theta\rangle_\mu^{1/2}h-\widetilde{\partial}_i\big((T^J\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j)(\widetilde{\partial}_jV^i)+T^J\langle\partial_\theta\rangle_\mu^{1/2}V^i\big)\big)||_{L^2(\Omega)}$$

$$\leq C_s||\widetilde{x}||_{H^{s+1}(\Omega)}\big(||V||_{H^{(s-1,1/2)}(\Omega)}+P(||h||_{s-1})||\widetilde{\partial}h||_{H^s(\Omega)}\big), \quad (2.11.6)$$

$$||\gamma^{ij}D_t\langle\partial_\theta\rangle_\mu^{1/2}T^IV_j+(\langle\partial_\theta\rangle_\mu^{1/2}T^I\gamma^{ij})\widetilde{\partial}_ih||_{L^2(\partial\Omega)}$$

$$\leq C_s'(||\mathcal{T}\widetilde{x}||_{H^{s+1/2}(\partial\Omega)}+1)||\widetilde{\partial}h||_{H^{s+1/2}(\partial\Omega)}. \quad (2.11.7)$$

We recall that the terms on the left-hand sides of (2.11.5)-(2.11.6) are needed to control $\mathcal{E}^s$. We will eventually show that $\mathcal{E}^s$ controls $||V||_{H^{(s,1/2)}(\Omega)}$, $||\widetilde{\partial}h||_{H^s(\Omega)}$ and $||\widetilde{x}||_{H^{s+1}(\Omega)}$. Similarly we will use the estimate (2.11.7) to control $\mathcal{E}^s_\varepsilon$ and we will eventually show that this controls $\varepsilon^{-1}||V||_{H^{s+1}(\Omega)}$ and $\varepsilon^{-1}||\widetilde{\partial}h||_{H^{s+1}(\Omega)}$.

The terms on the right-hand side of (2.11.7) are higher-order and to deal with them we need to use tangential smoothing, which introduces a term behaving like ε^{-1}. Since we will estimate $\varepsilon\sqrt{\mathcal{E}^s_\varepsilon}$ this will not cause issues.

The reason we do not commute $\langle\partial_\theta\rangle_\mu^{1/2}$ all the way through on the left-hand side of (2.11.6) is that it would generate an error term involving $||V||_{H^{s+1}(\Omega)}$ which can only be controlled in terms of $\varepsilon^{-1}\sqrt{\mathcal{E}^s_\varepsilon}$, which would not allow us to close the energy estimates in the next sections on a time interval independent of ε.

Proof. We start by noting that if V, h satisfy (2.11.1)- (2.11.2), then all of the quantities on the right-hand sides of (2.11.5)-(2.11.7) are finite. Therefore, by an approximation argument it suffices to prove this result assuming that V, h are smooth. We first show that

$$||\langle\partial_\theta\rangle_\mu^{1/2}(D_t T^I V - \widetilde{\partial}((T^I\widetilde{x}^j)(\widetilde{\partial}_j h)) + T^I h)||_{L^2}$$

$$\leq C_s||\widetilde{x}||_{H^{s+1}}(||\widetilde{\partial}h||_{H^s} + ||\widetilde{\partial}h||_{H^2}) + ||\langle\partial_\theta\rangle_\mu^{1/2}T^I\widetilde{\partial}\phi(h)||_{L^2}, \quad (2.11.8)$$

where $H^s = H^s(\Omega)$ and $L^2 = L^2(\Omega)$. Note that by Lemma 2.A.1 we have:

$$||[T^I, \langle\partial_\theta\rangle_\mu^{1/2}]D_t V||_{L^2(\Omega)} \leq C||D_t V||_{H^s(\Omega)} = C||\widetilde{\partial}h||_{H^s(\Omega)}, \quad (2.11.9)$$

$$||\widetilde{\partial}[T^I, \langle\partial_\theta\rangle_\mu^{1/2}]\widetilde{x}||_{L^2(\Omega)} \leq C(M)||\widetilde{x}||_{H^{s+1}(\Omega)}, \quad (2.11.10)$$

for $|I| \leq s$, and so combining this with (2.11.8) gives (2.11.5). To prove (2.11.8), we start by computing $T^I(D_t V + \widetilde{\partial}h)$. The vector fields T^I commute with D_t and so we just need to compute $T^I\widetilde{\partial}h$. Using (2.D.2), we have:

$$T^I\widetilde{\partial}h - \widetilde{\partial}\big(T^I h - T^I\widetilde{x}^j(\widetilde{\partial}_j h)\big) = (T^I\widetilde{x}^j)\widetilde{\partial}\widetilde{\partial}_j h + \sum(\widetilde{\partial}T^{I_1}\widetilde{x})\cdots(\widetilde{\partial}T^{I_\ell}\widetilde{x})(T^{I_{\ell+1}}\widetilde{\partial}h), \quad (2.11.11)$$

where the sum is taken over all indices with $|I_1| + ... + |I_{\ell+1}| \leq |I|$ so that $|I_j| \leq s-1$ for $j \leq \ell$ and $|I_{\ell+1}| \geq 1$.

To control the first term on the right-hand side of (2.11.11) we apply the fractional product rule (2.A.8):

$$||\langle\partial_\theta\rangle_\mu^{1/2}(T^I\widetilde{x}\cdot\widetilde{\partial}^2 h)||_{L^2(\Omega)} \leq C||T^I\widetilde{x}||_{H^2(\Omega)}(||\widetilde{\partial}^2 h||_{L^2(\Omega)} + ||\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{\partial}^2 h||_{L^2(\Omega)})$$

$$\leq C(M)||\widetilde{x}||_{H^5(\Omega)}||\widetilde{\partial}h||_{H^{(1,1/2)}(\Omega)}, \quad (2.11.12)$$

for $|I| = s \leq 3$ as required. If instead $|I| = s \geq 4$, we use the fractional product rule (2.A.8) to bound:

$$||\langle\partial_\theta\rangle_\mu^{1/2}(T^I\widetilde{x}\cdot\widetilde{\partial}^2 h)||_{L^2(\Omega)} \leq C||\langle\partial_\theta\rangle_\mu^{1/2}T^I\widetilde{x}||_{L^2(\Omega)}||\widetilde{\partial}h||_{H^3(\Omega)}, \tag{2.11.13}$$

and this is also bounded by the right-hand side of (2.11.5). It just remains to bound the terms in the sum in (2.11.11). Suppose for each $j \leq \ell$, $|I_j| \leq 2$. By the Leibniz rule (2.A.8), we have:

$$||\langle\partial_\theta\rangle_\mu^{1/2}\big((\widetilde{\partial}T^{I_1}\widetilde{x})\cdots(\widetilde{\partial}T^{I_\ell}\widetilde{x})(T^{I_{\ell+1}}\widetilde{\partial}h)\big)||_{L^2(\Omega)}$$
$$\leq C||(\widetilde{\partial}T^{I_1}\widetilde{x})\cdots(\widetilde{\partial}T^{I_\ell}\widetilde{x})||_{H^2(\Omega)}||\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{\partial}h||_{L^2(\Omega)}. \tag{2.11.14}$$

Since $H^2(\Omega)$ is an algebra, the first factor is bounded by $C(M)||\widetilde{x}||_{H^5(\Omega)}^{\ell}$. This just leaves the case that there is at least one $j \leq \ell$ with $|I_j| \geq 3$. However note that in this case we must have that $s \geq 4$ and $|I_{j'}| \leq s-4$ for $j' \leq \ell+1$, $j' \neq j$, and so using (2.A.8) and the algebra property of $H^2(\Omega)$ we have:

$$||\langle\partial_\theta\rangle_\mu^{1/2}\big((\widetilde{\partial}T^{I_1}\widetilde{x})\cdots(\widetilde{\partial}T^{I_\ell}\widetilde{x})(T^{I_{\ell+1}}\widetilde{\partial}h)\big)||_{L^2(\Omega)}$$
$$\leq C(M)||\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{\partial}T^{I_j}\widetilde{x}||_{L^2(\Omega)}||\widetilde{x}||_{H^{s-1}(\Omega)}^{\ell-1}||\widetilde{\partial}h||_{H^{s-2}(\Omega)}. \tag{2.11.15}$$

The first factor is bounded by $||\widetilde{x}||_{H^{s+1}(\Omega)}$ since $I_j \leq s-1$, and this completes the proof of (2.11.8).

The estimate (2.11.6) is similar. We will actually prove the slightly stronger estimate:

$$||\partial_y\big(e'(h)D_tT^Jh - (\widetilde{\partial}_iT^J\widetilde{x}^j)(\widetilde{\partial}_jS_\varepsilon V^i) + \operatorname{div} T^JV\big)||_{L^2(\Omega)}$$
$$\leq C_s\big(||\widetilde{x}||_{H^s(\Omega)}+1\big)\big(||\widetilde{\partial}h||_{H^s(\Omega)}+P(||\widetilde{\partial}h||_{H^{s-1}(\Omega)})\big), \tag{2.11.16}$$

which implies (2.11.6) since $||\langle\partial_\theta\rangle_\mu^{1/2}f||_{L^2(\Omega)} \leq ||f||_{H^1(\Omega)}$. We apply $\partial_y T^J$ to $e'(h)D_th + \text{div} V = 0$. We start with

$$||\partial_y T^J(e'(h)D_th) - \partial_y(e'(h)D_tT^Jh)||_{L^2(\Omega)}$$
$$\leq C_s P(||\widetilde{\partial}h||_{H^{s-2}(\Omega)})(||\widetilde{\partial}h||_{H^{s-1}(\Omega)} + ||D_th||_{H^{s-1}(\Omega)}), \quad (2.11.17)$$

which follows from a straightforward modification of the proof of Lemma 2.D.7.

It just remains to control $[T^J, \text{div}]V$. We start by writing:

$$T^J\widetilde{\partial}_i V^i = \widetilde{\partial}_i T^J V^i - (\widetilde{\partial}_i T^J \widetilde{x}^j)(\widetilde{\partial}_j V^i) + \sum (\widetilde{\partial} T^{J_1}\widetilde{x}) \cdots (\widetilde{\partial} T^{J_\ell}\widetilde{x})(T^{J_{\ell+1}}\widetilde{\partial} V), \quad (2.11.18)$$

where the sum is over all $J_1 + \cdots J_{\ell+1} = J$ with $|J_{\ell+1}| \geq 1$ and $|J_j| \leq |J| - 1 = s - 2$ for each $j \leq \ell + 1$. Applying ∂_y, it suffices to control the L^2 norms of:

$$(\widetilde{\partial} T^{J_1}\widetilde{x})(\widetilde{\partial} T^{J_2}\widetilde{x}) \cdots (\partial_y T^{J_{\ell+1}}\widetilde{\partial} V), \text{ and } (\partial_y \widetilde{\partial} T^{J_1}\widetilde{x})(\widetilde{\partial} T^{J_2}\widetilde{x}) \cdots (T^{J_{\ell+1}}\widetilde{\partial} V). \quad (2.11.19)$$

Estimates for these terms can be obtained in the same way as the estimates we used above to control the sum in (2.11.11). To control the first term in (2.11.19), if $|I_j| \leq 2$ for each $j \leq \ell$, then we use Sobolev embedding:

$$||(\widetilde{\partial} T^{J_1}\widetilde{x})(\widetilde{\partial} T^{J_2}\widetilde{x}) \cdots (T^{J_{\ell+1}} V)||_{L^2(\Omega)}$$
$$\leq C||(\widetilde{\partial} T^{J_1}\widetilde{x})(\widetilde{\partial} T^{J_2}\widetilde{x}) \cdots (\widetilde{\partial} T^{J_\ell}\widetilde{x})||_{H^2(\Omega)} ||\partial_y T^{J_{\ell+1}}\widetilde{\partial} V||_{L^2(\Omega)}, \quad (2.11.20)$$

and the first factor here is bounded by $C(M')$ using the fact that $H^2(\Omega)$ is an algebra. To control the second factor, we write it as $||\partial_y T^{J_{\ell+1}}(u \cdot \partial_y V)||_{L^2(\Omega)}$ and then note that since $|J_{\ell+1}| \leq s - 2$, we can bound $||\partial_y T^{J_{\ell+1}}(u \cdot \partial_y V)||_{L^2(\Omega)} \leq C(M)||\widetilde{x}||_{H^s(\Omega)}||V||_{H^s(\Omega)}$ by using similar arguments to the above. If there is a multi-index I_j with $|I_j| \geq 3$ then this forces

$|I_{j'}| \leq s-4$ for each $j' \neq j$ and so we put all the factors except $\widetilde{\partial} T^{I_j}\widetilde{x}$ into L^∞, apply Sobolev embedding and argue as above. To control the first type of term from (2.11.19) is similar, noting that in this case there are no more than $2+(s-2)$ derivatives falling on $\widetilde{x}$ at any point.

We now prove (2.11.7). We apply T^I to $\gamma^{ij} D_t V_i = \gamma^{ij}\widetilde{\partial}_i h = 0$ and we have:

$$0 = (T^I \gamma^{ij})\widetilde{\partial}_i h + \gamma^{ij} T^I \widetilde{\partial}_i h + \sum (T^{I_1}\gamma^{ij}) \cdots (T^{I_\ell}\gamma^{ij})(T^{I_{\ell+1}}\widetilde{\partial}_i h), \tag{2.11.21}$$

where $|I_1| + \cdots |I_{\ell+1}| \leq s, |I_{\ell+1}| \geq 1, |I_j| \leq s-1, j \leq \ell+1$.

We now recall that $\gamma^{ij} = \gamma^{ab} A^i{}_a A^j{}_b$ where $\gamma^{ab} = \delta^{ab} - N^a N^b$ and in particular γ^{ab} is independent of $\widetilde{x}$. Applying (2.D.2) repeatedly, it therefore suffices to control the $L^2(\partial\Omega)$ norms of:

$$\langle \partial_\theta \rangle_\mu^{1/2} (\gamma^{ij} T^I \widetilde{\partial}_i h), \quad \langle \partial_\theta \rangle_\mu^{1/2} \big((\widetilde{\partial} T^I \widetilde{x})(\widetilde{\partial} h) \big), \quad \langle \partial_\theta \rangle_\mu^{1/2} \big((\widetilde{\partial} T^{I_1} \widetilde{x}) \cdots (\widetilde{\partial} T^{I_\ell} \widetilde{x})(T^{I_{\ell+1}} \widetilde{\partial} h) \big), \tag{2.11.22}$$

with the same conditions on the multi-indices $I_1, ..., I_{\ell+1}$ as above.

To deal with the first term in (2.11.22), we just use the Leibniz rule (2.A.8) and control it by:

$$||\gamma^{ij}||_{H^2(\partial\Omega)} ||\langle \partial_\theta \rangle_\mu^{1/2} T^I \widetilde{\partial} h||_{L^2(\partial\Omega)} \leq C(M) ||\widetilde{x}||_{H^3(\partial\Omega)} ||\langle \partial_\theta \rangle_\mu^{1/2} T^I \widetilde{\partial} h||_{L^2(\partial\Omega)}, \tag{2.11.23}$$

since γ is quadratic in A, where we have use the fact that $H^2(\partial\Omega)$ is an algebra. By the trace inequality this is controlled by the right-hand side of (2.11.7). To deal with the second term in (2.11.22), when $|I| = s \leq 2$ we use the Leibniz rule (2.A.8) and control it by $||\widetilde{\partial} T^I \widetilde{x}||_{H^2(\partial\Omega)} ||\widetilde{\partial} h||_{H^{1/2}(\partial\Omega)}$, and this first factor is controlled by $C(M)||\widetilde{x}||_{H^5(\partial\Omega)} \leq C(M)||\widetilde{x}||_{H^6(\Omega)}$ by the trace inequality. If $|I| = s \geq 3$ we instead control it by $||\langle \partial_\theta \rangle_\mu^{1/2} \widetilde{\partial} T^I \widetilde{x}||_{L^2(\partial\Omega)} ||\widetilde{\partial} h||_{H^2(\partial\Omega)}$, which is bounded by $C(M)||\mathcal{T}\widetilde{x}||_{H^{s+1/2}(\partial\Omega)}$ times $||\widetilde{\partial} h||_{H^s(\partial\Omega)}$.

Estimates for the third term in (2.11.22) can be obtained in a similar fashion. If $|I_j| \leq 2$ for each $j \leq \ell$, then

$$||\langle\partial_\theta\rangle_\mu^{1/2}\big((\widetilde{\partial}T^{I_1}\widetilde{x})\cdots(\widetilde{\partial}T^{I_\ell}\widetilde{x})(T^{I_{\ell+1}}\widetilde{\partial}h)||_{L^2(\partial\Omega)} \leq C(M)||\widetilde{x}||^\ell_{H^5(\partial\Omega)}||\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{\partial}h||_{H^{s-1}(\partial\Omega)},$$
(2.11.24)

and by the trace inequality this is bounded by the right-hand side of (2.11.7). If instead $|I_j| \geq 3$ for some $j \leq \ell$ then this forces $s \geq 4$ and $|I_{j'}| \leq s-4$ for $j' \neq j$ and so the result is bounded by:

$$||\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{\partial}T^{I_j}\widetilde{x}||_{L^2(\partial\Omega)}||\widetilde{x}||^{\ell-1}_{H^s(\partial\Omega)}||\widetilde{\partial}h||_{H^{s-1}(\partial\Omega)} \leq C_s||\mathcal{T}\widetilde{x}||_{H^{s-1/2}(\partial\Omega)}||\widetilde{\partial}h||_{H^{s-1}(\partial\Omega)}. \qquad \square$$

2.12 Uniform energy estimates for the smoothed problem up to a fixed time

We define:

$$\mathcal{E}^s = K^s + \textstyle\sum_{\mu=0}^N \sum_{|I|\leq s}\mathcal{E}^I_\mu, \qquad \mathcal{E}^s_\varepsilon = K^s_\varepsilon + \sum_{|I|\leq s}\mathcal{E}^I_\varepsilon, \tag{2.12.1}$$

where, with $\langle\partial_\theta\rangle_\mu^{1/2}$ defined by (2.3.10):

$$K^s = \textstyle\sum_{\mu=0}^N ||\operatorname{curl}\langle\partial_\theta\rangle_\mu^{1/2}V||^2_{H^{s-1}(\Omega)}, \quad K^s_\varepsilon = ||\operatorname{curl}\partial V||^2_{H^{s-1}(\Omega)} + ||\operatorname{div}\partial V||^2_{H^{s-1}(\Omega)}, \tag{2.12.2}$$

and, with notation as in Sections 2.3.1 and 2.4:

$$\mathcal{E}^I_\mu = \int_\Omega \delta_{ij}(T^I\langle\partial_\theta\rangle_\mu^{1/2}V^i)(T^I\langle\partial_\theta\rangle_\mu^{1/2}V^j) + e'(h)|T^I\langle\partial_\theta\rangle_\mu^{1/2}h|^2\,\widetilde{\kappa}dy$$
$$+\int_{\partial\Omega} N_iN_j(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^i)(T^I\langle\partial_\theta\rangle_\mu^{1/2}J_\varepsilon x^j)\,|\widetilde{\partial}h|\widetilde{\nu}dS, \tag{2.12.3}$$

where $\widetilde{\nu}dS$ is the Eulerian surface measure, and

$$\mathcal{E}^I_\varepsilon = \int_{\partial\Omega}\gamma^{ij}(T^IV_i)(T^IV_j)\,\widetilde{\nu}dS, \qquad T^I \in \mathcal{T}^s. \tag{2.12.4}$$

To control h, we will use:

$$\mathcal{W}^s = \sum_{k\leq s} \int_\Omega e'(h)|D_t^{k+1}h|^2 + |D_t^k\widetilde{\partial}h|^2\,\widetilde{\kappa}dy, \tag{2.12.5}$$

and to control $\widetilde{x}$ we will use:

$$\mathcal{A}^s = ||\,\mathrm{div}\,\partial_y J_\varepsilon x||^2_{H^{s-1}(\Omega)} + ||\,\mathrm{curl}\,\partial_y J_\varepsilon x||^2_{H^{s-1}(\Omega)}. \tag{2.12.6}$$

The energy we consider is then:

$$\mathscr{E}^s = \mathcal{A}^s + \mathcal{W}^s + \mathcal{E}^s + \varepsilon^2 \mathcal{E}^s_\varepsilon. \tag{2.12.7}$$

We will also write $\mathscr{E}^s_0$ for the quantity $\mathscr{E}^s$ with V replaced by V_0^ε, x replaced with x_0 and $D_t^{k+1}h$ replaced by h^ε_{k+1}, with $V_0^\varepsilon, h^\varepsilon_{k+1}$ defined by in Section 2.4.3. The goal of this section is to prove the following theorem:

Theorem 2.12.1. *Suppose that the initial data $(V_0^\varepsilon, h_0^\varepsilon)$ are such that $\mathscr{E}^s_0 < \infty$ for some $s \geq 1$. There is a positive, continuous function $\mathcal{F}_s$ so that the following holds: If $V \in \mathcal{X}^{s+1}(T)$ is a solution to the smoothed Euler's equations (2.4.7)-(2.4.11) so that $(V,h)|_{t=0} = (V_0^\varepsilon, h_0^\varepsilon)$ and the a priori assumptions (2.10.8)-(2.10.9) hold, then:*

$$\mathscr{E}^s(t) \leq \mathcal{F}_s\left(M', L, \delta^{-1}, \mathscr{E}_0^{s-1}\right)\mathscr{E}_0^s, \qquad 0 \leq t \leq T. \tag{2.12.8}$$

We now take $M_0', L_0, \delta_0 > 0$ so that:

$$|\partial x_0/\partial y| + |\partial y/\partial x_0| + \sum_{k+|I|\leq 3} |\partial_y^{|I|} V_k^\varepsilon| + ||\widetilde{x}_0||_{H^6(\Omega)} \leq M_0', \tag{2.12.9}$$

$$\sum_{k+|I|\leq 3} |\partial_y^{|I|}\widetilde{\partial} h_k^\varepsilon| + |h_k^\varepsilon| \leq L_0, \tag{2.12.10}$$

$$-\widetilde{\partial}_{N_0} h_0|_{\partial\Omega} \geq \delta_0, \tag{2.12.11}$$

where we are writing $\widetilde{\partial}_{N_0} = N_0^i \widetilde{\partial}_i$ with N_0^i the unit normal to $\partial\Omega$ with respect to the metric $\widetilde{g}$

at $t = 0$.

We will show that the above energy estimate implies:

Corollary 2.12.1. *Let $r \geq 7$. There are continuous, strictly positive functions $\mathscr{T}_r, \mathscr{C}_r, \mathscr{C}_r'$ with $\mathscr{C}_r, \mathscr{C}_r'$ depending on $M_0', L_0, \delta_0^{-1}, \mathscr{E}_0^{r-1}$ so that if*

$$T \leq \mathscr{T}_r(M_0', L_0, \mathscr{E}_0^{r-1}, \delta_0^{-1}),$$

and $V \in \mathcal{X}^{r+1}(T)$ satisfies the smoothed-out Euler equations (2.4.11) with initial data satisfying (2.12.11), then:

$$\mathscr{E}^{r-1}(t) \leq \mathscr{C}_r \mathscr{E}_0^{r-1}, \qquad 0 \leq t \leq T, \tag{2.12.12}$$

and with $H^s = H^s(\Omega)$

$$||V(t)||^2_{H^{(r-1,1/2)}} + ||J_\varepsilon x(t)||^2_{H^r} + ||\widetilde{\partial} h(t)||^2_{r-1} + \varepsilon^2(||V(t)||^2_{H^r} + ||\widetilde{\partial} h(t)||^2_{H^r}) \leq \mathscr{C}_r' \mathscr{E}_0^{r-1}, \tag{2.12.13}$$

for $0 \leq t \leq T$.

Before proving Theorem 2.12.1, we collect a few preliminary results. In Lemma 2.12.1, we show that we control $\widetilde{x}$, V and h provided we control $\mathcal{A}^s$, $\mathcal{W}^s$ and the energies $\mathcal{E}^s$. In Lemma 2.12.2 and Corollary 2.12.2, we show that we control $\mathcal{A}^s$ and $\mathcal{W}^s$ provided that we control $\mathscr{E}^s$.

Lemma 2.12.1. *Fix $r \geq 7$and suppose that $V \in \mathcal{X}^{r+1}(T)$ satisfies the smoothed-out Euler equations (2.4.11) and that the apriori assumptions (2.5.2),(2.6.6) and the Taylor sign condition (2.1.9) hold. For each $0 \leq s \leq r-1$, there is a positive, continuous function $C_s = C_s(M', L, \delta^{-1}, \mathcal{A}^{s-1}, \mathcal{W}^{s-1}, \mathcal{E}^{s-1})$ so that the following estimates hold:*

$$||\widetilde{x}||^2_{H^{s+1}(\Omega)} + ||V||^2_{H^{(s,1/2)}(\Omega)} + ||V||^2_{\mathcal{X}^{s+1}} + ||\widetilde{\partial} h||^2_s + ||D_t h||^2_s \leq C_s\big(\mathcal{A}^s + \mathcal{W}^s + (1+\delta^{-1})\mathcal{E}^s\big), \tag{2.12.14}$$

and for $\mu = 0, ..., N$, we have:

$$||J_\varepsilon x||^2_{H^{s+1}(\Omega)} + ||\langle\partial_\theta\rangle_\mu^{1/2} V||^2_{H^s(\Omega)} + ||\langle\partial_\theta\rangle_\mu^{1/2} T^s \widetilde{\partial}\phi(h)||_{L^2(\Omega)} \leq C_s\big(\mathcal{A}^s + \mathcal{W}^s + (1+\delta^{-1})\mathcal{E}^s\big). \tag{2.12.15}$$

Finally:

$$||V||^2_{H^{s+1}(\Omega)} + ||\widetilde{\partial} h||^2_{s+1} \leq C_s(\mathcal{A}^s + \mathcal{W}^s + (1+\delta^{-1})\mathcal{E}^s + \varepsilon^{-2}\mathcal{E}^s_\varepsilon). \tag{2.12.16}$$

Proof. The first estimate in (2.12.14) follows from the first estimate in (2.12.15), since $\widetilde{x} = S_\varepsilon x = J_\varepsilon^2 x$ and J_ε is bounded on Sobolev spaces. The second estimate in (2.12.14) follows after summing the second estimate in (2.12.15) over all $\mu = 0, ..., N$ and using Lemma 2.A.20. To prove the third estimate we note that if V solves the smoothed problem (2.4.11) then $||V||_{\mathcal{X}^{s+1}} \leq ||V||_{H^s(\Omega)} + ||\widetilde{\partial} h||_s + ||\widetilde{\partial}\phi||_s$. Using Theorem 2.7.1 to control $||\widetilde{\partial}\phi||_s$, this estimate then follows from the estimate for $||\widetilde{\partial} h||_s$. To prove the estimate for $||\widetilde{\partial} h||_s$ and $||D_t h||_s$, we argue as in the proof of (2.6.13) and suppose that (2.12.14) holds for $s = 0, ..., m-1$. By definition $||D_t^m \widetilde{\partial} h||^2_{L^2(\Omega)} + ||D_t^{m+1} h||^2_{L^2(\Omega)} \leq C\mathcal{W}^s$ so we now suppose that $||D_t^k \widetilde{\partial} h||^2_{H^\ell(\Omega)} + ||D_t^{k+1} h||^2_{H^\ell(\Omega)}$ is bounded by the right-hand side of (2.12.14) for $k + \ell = s$ and some $\ell \geq 0$. By induction it suffices to prove that $||D_t^{k-1}\widetilde{\partial} h||^2_{H^{\ell+1}(\Omega)} + ||D_t^{k-1} D_t h||^2_{H^{\ell+1}(\Omega)}$ is bounded by the right-hand side of (2.12.14). Writing $\partial_a D_t^{k-1} D_t h = A^i{}_a \widetilde{\partial}_i D_t^{k-1} D_t h$ and then $\widetilde{\partial} D_t^{k-1} D_t h = D_t^k \widetilde{\partial} h + [\widetilde{\partial}, D_t^k] h$, using (2.D.23) to handle the commutator and (2.A.45):

$$\begin{aligned} &||\partial_y D_t^{k-1} D_t h||_{H^\ell(\Omega)} \\ &\qquad \leq C(M', ||\widetilde{x}||_{H^\ell(\Omega)}, ||V||_{\mathcal{X}^{m-1}})\big(||D_t^k \widetilde{\partial} h||_{H^\ell(\Omega)} + (||V||_{\mathcal{X}^m} + ||\widetilde{x}||_{H^{\ell+1}(\Omega)})||\widetilde{\partial} h||_{m-1}\big). \end{aligned} \tag{2.12.17}$$

When $\ell = 0$ then by the inductive assumption and the definition of the energy $\mathcal{W}$, all of the terms on the right-hand side are bounded by the right-hand side of (2.12.14). The estimate

(2.12.14) for $s = m$ now follows from the inductive assumption and the following estimate, which we claim holds whenever $k + \ell = m, \ell \geq 1$:

$$||D_t^k \widetilde{\partial} h||^2_{H^\ell(\Omega)} \leq C\big(||D_t^{k+2}h||_{H^{\ell-1}} + (||\widetilde{x}||_{H^{m+1}} + ||V||_{H^m} + ||V||_{\mathcal{X}^m})(||\widetilde{\partial} h||_{m-1} + ||D_t h||_{m-1})\big), \tag{2.12.18}$$

where $C = C(M, ||\widetilde{x}||_{H^m(\Omega)}, ||V||_{\mathcal{X}^m})$ and $H^s = H^s(\Omega)$. This estimate follows directly from the elliptic estimate (2.5.9), the fact that $D_t^k \widetilde{\Delta} h = D_t^k\big(e'(h) D_t^2 h - (\widetilde{\partial}_i S_\varepsilon V^j)(\widetilde{\partial}_j V^i)\big)$ and Lemmas 2.D.9 and 2.D.7 to control these.

We now prove the first estimate in (2.12.15). When $s \leq 6$ there is nothing to prove since $||J_\varepsilon x||_{H^6(\Omega)} \leq M'$, so we assume $s \geq 6$. In fact the below argument works provided $s \geq 2$ and this assumption is only needed to ensure that the trace map is continuous. The point of the below manipulations is to replace the derivative ∂_y with tangential vector fields T. Using (2.B.91), we have that:

$$||\partial_y J_\varepsilon x||^2_{H^s(\Omega)} \leq C_s\big(\mathcal{A}_s + ||\partial_y J_\varepsilon x||^2_{H^{s-1/2}(\partial\Omega)} + ||J_\varepsilon x||^2_{H^1(\Omega)}\big), \tag{2.12.19}$$

with $C_s = C_s(M', ||\widetilde{x}||_{H^s(\Omega)})$. To control the boundary term here it suffices to control $||T\partial_y J_\varepsilon x||_{H^{s-3/2}(\partial\Omega)}$ for $s \geq 2$ and any $T \in \mathcal{T}$, and by the trace inequality (2.A.41), this is under control if we control $||TJ_\varepsilon x||_{H^s(\Omega)}$. Finally, we note that because of the boundary term in the energy, for each $T \in \mathcal{T}$ we have:

$$||TJ_\varepsilon x||^2_{H^s(\Omega)} \leq C_s\big(||\operatorname{div} TJ_\varepsilon x||^2_{H^{s-1}(\Omega)} + ||\operatorname{curl} TJ_\varepsilon x||^2_{H^{s-1}(\Omega)} + \delta^{-1}\mathcal{E}_s\big), \tag{2.12.20}$$

again with $C_s = C_s(M', ||\widetilde{x}||_{H^s(\Omega)})$. The first and second terms here are bounded by $\mathcal{A}^s$ and using induction and the first estimate in (2.12.14), this implies the first estimate in (2.12.15).

To prove the second estimate in (2.12.15), we note that by the elliptic estimate (2.B.16), we

have:

$$||\langle\partial_\theta\rangle_\mu^{1/2} V||^2_{H^s(\Omega)}$$

$$\leq C(M', ||\widetilde{x}||_{H^s(\Omega)})\big(||\operatorname{div}\langle\partial_\theta\rangle_\mu^{1/2} V||^2_{H^{s-1}(\Omega)} + ||\operatorname{curl}\langle\partial_\theta\rangle_\mu^{1/2} V||^2_{H^{s-1}(\Omega)} + \mathcal{E}^s\big). \quad (2.12.21)$$

The last two terms are controlled by the right-hand side of (2.12.15). For the first term, we use Lemma 2.A.1:

$$||[\operatorname{div}, \langle\partial_\theta\rangle_\mu^{1/2}]V||_{H^{s-1}(\Omega)} \leq C(M', ||\widetilde{x}||_{H^s(\Omega)})\big(||\widetilde{x}||_{H^{s+1}(\Omega)} + ||V||_{H^s(\Omega)}\big), \quad (2.12.22)$$

and so using $\operatorname{div} V = -e'(h)D_t h$, it just remains to bound $||\langle\partial_\theta\rangle_\mu^{1/2}(e'(h)D_t h)||_{H^{s-1}(\Omega)}$. We first bound this by $||e'(h)D_t h||_{H^s(\Omega)}$ and then using induction and Lemma 2.D.7, this is controlled by $C(M', L, \mathcal{W}^{s-1})||D_t h||_{H^s(\Omega)}$. We write $\partial_y^I = \partial_y^J(u\cdot\widetilde{\partial})$, where $|I| = s$, $|J| = s-1$, then apply (2.A.45) and the commutator estimate (2.A.10) to control this by $C_s||\widetilde{\partial}h||_s$. Since $\rho = \rho(h)$, the third estimate in (2.12.15) is a consequence of Theorem 2.7.6, (2.12.14).

The estimate (2.12.16) follows from the definition of $\mathcal{E}^s_\varepsilon$, the elliptic estimate (2.5.11) and (2.12.14). $\square$

We now control the energy for the wave equation $\mathcal{W}^s$ in terms of $\mathcal{E}^s, \mathcal{A}^s$:

Lemma 2.12.2. *With the same hypotheses as Lemma 2.12.1, there is a constant C_s' depending on M', L, δ^{-1}, T, $\sup_{0\leq t\leq T}\mathcal{A}^{s-1}(t) + \mathcal{E}^{s-1}(t)$ and $\mathcal{W}^{s-1}(0)$ so that:*

$$\mathcal{W}^s(t) \leq C_s'\Big(\mathcal{W}^s(0) + \int_0^t \mathcal{A}^s(\tau) + \mathcal{E}^s(\tau)\,d\tau\Big), \quad 0 \leq t \leq T. \quad (2.12.23)$$

Proof. By Lemma 2.6.2, writing $\mathcal{F} = -(\widetilde{\partial}_i S_\varepsilon V^j)(\widetilde{\partial}_i V^i)$, we have:

$$\frac{d}{dt}\sqrt{\mathcal{W}^s} \leq C_s^1\big(||\mathcal{F}||_{s,0} + ||\mathcal{F}||_{s-1} + ||V||_{\mathcal{X}^{s+1}} + P(L, ||\widetilde{\partial}h||_{s-1}, \sqrt{\mathcal{W}^{s-1}})\big), \quad (2.12.24)$$

with $C_s^1 = C_s^1(M', L, T, ||\widetilde{x}||_{H^s(\Omega)}, ||V||_{\mathcal{X}^s})$. Using Lemma 2.D.9 to control $\mathcal{F}$ and Lemma 2.12.1 to control V, $\widetilde{x}$ and $\widetilde{\partial} h$ in terms of $\mathcal{A}, \mathcal{W}$ and $\mathcal{E}$, this implies:

$$\frac{d}{dt}\sqrt{\mathcal{W}^s} \leq C_s^2\big(M', L, \delta^{-1}, T, \sqrt{\mathcal{W}^{s-1}}, \sqrt{\mathcal{A}^{s-1}}, \sqrt{\mathcal{E}^{s-1}}\big)\big(\sqrt{\mathcal{W}^s} + \sqrt{\mathcal{A}^s} + \sqrt{\mathcal{E}^s}\big). \tag{2.12.25}$$

Multiplying by the integrating factor $e^{-tC_s^2}$, integrating from 0 to T and using induction gives (2.12.23). $\square$

We will need the following estimate to control $\widetilde{x}$:

Lemma 2.12.3. *For $s \geq 0$, there are constants C_s' depending on M', L, δ^{-1}, T and $\sup_{0\leq t\leq T}\big(\mathcal{A}^{s-1}(t) + \mathcal{E}^{s-1}(t)\big)$ so that if (2.11.3) is satisfied, then for $0 \leq t \leq T$:*

$$||D_t \operatorname{div} J_\varepsilon \partial_y x||^2_{H^{s-1}(\Omega)}$$

$$\leq C_s'\big(||\operatorname{div} V||^2_{H^s(\Omega)} + ||\big(J_\varepsilon \operatorname{div} \partial_y V - \operatorname{div} J_\varepsilon \partial_y V\big)||^2_{H^{s-1}(\Omega)} + \mathcal{A}^s + \mathcal{E}^s\big). \tag{2.12.26}$$

In addition, for any multi-index I with $|I| = s-1$ and $\mu = 0, ..., N$ there is a two-form $R = R^I_{ij}$ with $||R||_{L^2(\Omega)} \leq C_s'(\mathcal{A}^s + \mathcal{E}^s)$ so that for $0 \leq t \leq T$:

$$||D_t^2 \partial_y^I (\operatorname{curl} \partial_y J_\varepsilon x) - D_t R^I||^2_{L^2(\Omega)}$$

$$\leq C_s'\big(||\operatorname{curl} D_t V||^2_{H^s(\Omega)} + ||J_\varepsilon \operatorname{curl} \partial_y V - \operatorname{curl} J_\varepsilon \partial_y V||^2_{H^{s-1}(\Omega)} + \mathcal{A}^s + \mathcal{E}^s\big). \tag{2.12.27}$$

Proof. We start by writing $D_t \operatorname{div} \partial_y J_\varepsilon x = -(\widetilde{\partial}_i S_\varepsilon V^j)\widetilde{\partial}_j \partial_y J_\varepsilon x^i + \operatorname{div} \partial_y J_\varepsilon V$. Applying $s-1$ derivatives to this expression, we first prove:

$$||\partial_y^J(\widetilde{\partial} S_\varepsilon V)\partial_y^K(\widetilde{\partial}\partial_y J_\varepsilon x)||_{L^2(\Omega)} \leq C_s'(\mathcal{A}^s + \mathcal{E}^s), \quad |J| + |K| = s-1. \tag{2.12.28}$$

When $|J| \leq 2$ we bound the first factor in L^∞ by M' and the second factor by $||\widetilde{\partial}\partial_y J_\varepsilon x||_{H^{s-1}(\Omega)} \leq$

$C(M)||J_\varepsilon x||_{H^{s+1}(\Omega)}$. If $|J| \geq 3$ then $|K| \leq s-4$ and so we bound the first factor in $L^2(\Omega)$ by $||V||_{H^s(\Omega)}$ and the second factor by $||\partial_y^K \widetilde{\partial}\partial_y J_\varepsilon x||_{L^\infty(\Omega)} \leq C||\widetilde{\partial}\partial_y J_\varepsilon x||_{H^{s-2}(\Omega)} \leq C(M)||\widetilde{x}||_{H^s(\Omega)}||J_\varepsilon x||_{H^s(\Omega)}$. By Lemma 2.12.1, we control all of these terms by the right-hand side of (2.12.26).

We now control $||\,\mathrm{div}\,\partial_y J_\varepsilon V||_{H^{s-1}(\Omega)}$. Noting that $||(J_\varepsilon \,\mathrm{div} - \mathrm{div}\, J_\varepsilon)V||_{H^{s-1}(\Omega)}$ appears on the right-hand side of (2.12.26), and that $[\partial_y, J_\varepsilon]$ and J_ε are bounded operators on $H^{s-1}(\Omega)$, it remains to control $||[\mathrm{div}, \partial_y]V||_{H^{s-1}(\Omega)}$. Writing $[\mathrm{div}, \partial_y]V = -(\partial_y A^a_{\ i})\partial_a V^i$ and arguing as above, we have $||[\mathrm{div}, \partial_y]V||_{H^{s-1}(\Omega)} \leq C(M)||\widetilde{x}||_{H^{s+1}(\Omega)}||V||_{H^s(\Omega)}$, and again using Lemma 2.12.1 this is bounded by the right-hand side of (2.12.26).

To prove (2.12.27), we start by writing:

$$
\begin{aligned}
D_t^2(\mathrm{curl}\,\partial_y J_\varepsilon x)_{ij} &= D_t\big(\big(-(\widetilde{\partial}_i S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon x_j + (\widetilde{\partial}_j S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon x_i\big) + (\mathrm{curl}\,\partial_y J_\varepsilon D_t x)_{ij}\big)\\
&= D_t\big((\widetilde{\partial}_j S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon x_i - (\widetilde{\partial}_i S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon x_j\big) - (\widetilde{\partial}_i S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon D_t x_j\\
&\quad + (\widetilde{\partial}_j S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon D_t x_i + (\mathrm{curl}\,\partial_y J_\varepsilon D_t^2 x)_{ij}. \qquad (2.12.29)
\end{aligned}
$$

The last two terms will be too high-order after we apply $s-1$ derivatives since we do not want an estimate that involves $||V||_{H^{s+1}(\Omega)}$. To handle this, for each of these terms, we write $(\widetilde{\partial} S_\varepsilon V)\cdot \widetilde{\partial}\partial_y D_t J_\varepsilon x = D_t(\widetilde{\partial} S_\varepsilon V \cdot \widetilde{\partial}\partial_y J_\varepsilon x) - (\widetilde{\partial} S_\varepsilon D_t V)\cdot \widetilde{\partial}\partial_y J_\varepsilon x + (\widetilde{\partial} S_\varepsilon V)\cdot(\widetilde{\partial} S_\varepsilon V)\cdot \widetilde{\partial}\partial_y J_\varepsilon x$. Writing:

$$
R_{ij} = 2\big(-(\widetilde{\partial}_i S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon x_j + (\widetilde{\partial}_j S_\varepsilon V^\ell)\widetilde{\partial}_\ell \partial_y J_\varepsilon x_i\big), \qquad (2.12.30)
$$

we have shown that for some constants $\alpha^{ijk\ell mn}, q^{ijk\ell}$:

$$
\begin{aligned}
D_t^2 \mathrm{curl}\,\partial_y J_\varepsilon x_{ij} - D_t R_{ij} = \mathrm{curl}\,\partial_y J_\varepsilon D_t^2 x_{ij} &+ \sum \alpha_{ij}^{k\ell mn}(\widetilde{\partial}_i S_\varepsilon V_j)(\widetilde{\partial}_k S_\varepsilon V_\ell)\widetilde{\partial}_m \partial_y J_\varepsilon x_n\\
&+ q_{ij}^{k\ell}(\widetilde{\partial}_i S_\varepsilon D_t V_j)\widetilde{\partial}_k \partial_y J_\varepsilon x_\ell, \qquad (2.12.31)
\end{aligned}
$$

For multi-index I with $|I|{=}s-1$, we define $R^I{=}\partial_y^I R$. The estimate for R^I follows exactly as above estimates.

Using (2.12.28), we have that R^I satisfies the stated estimate so it just remains to control the terms in the sum after applying ∂_y^I. To control the second term in the sum, we note that we also have (2.12.28) with V replaced by $D_t V = -\widetilde{\partial} h - \widetilde{\partial}\phi$, using the estimates in Lemma 2.12.1 for $\widetilde{\partial} h$.

To control the first term in the sum, we argue as in the proof of (2.12.28). If $|J| + |K| + |L| = s-1$ and either $|J|, |K| \le 2$ then

$$||\partial_y^J \widetilde{\partial} S_\varepsilon V||_{L^\infty(\Omega)} ||\partial_y^K \widetilde{\partial} S_\varepsilon V||_{L^\infty(\Omega)} ||\partial_y^L \widetilde{\partial} \partial_y J_\varepsilon x||_{L^2(\Omega)} \le C(M) ||J_\varepsilon x||_{H^{s+1}(\Omega)},$$

and if instead one of $|J|, |K| \ge 3$ then without loss of generality it is $|J|$ and then $|K|, |L| \le s-4$, so by Sobolev embedding,

$$||\partial_y^J \widetilde{\partial} S_\varepsilon V||_{L^2(\Omega)} ||\partial_y^K \widetilde{\partial} S_\varepsilon V||_{L^\infty(\Omega)} ||\partial_y^L \widetilde{\partial} \partial_y J_\varepsilon x||_{L^\infty(\Omega)} \le C(M) ||V||^2_{H^s(\Omega)} ||J_\varepsilon x||_{H^s(\Omega)},$$

as required. Finally, using the same arguments as above we can re-write $\partial_y^I \operatorname{curl} \partial_y J_\varepsilon D_t V$ in terms of $\partial_y^I \operatorname{curl} \partial_y D_t V$ and terms with L^2 norms bounded by the right-hand side of (2.12.27). Finally, we note that:

$$||\partial_y^I([\operatorname{curl}, \partial_y] D_t v)||_{L^2(\Omega)} \le C(M, ||\widetilde{x}||_{H^s(\Omega)}) ||\widetilde{x}||_{H^{s+1}(\Omega)} ||D_t V||_{H^s(\Omega)}, \tag{2.12.32}$$

which follows from the fact that $[\operatorname{curl}, \partial_y] D_t V_{ij} = (\partial_y A^a{}_j) \partial_a D_t V_i - (\partial_y A^a{}_j) \partial_a D_t V_j$ and using the arguments as above. Using the smoothed-out Euler's equations $D_t V = -\widetilde{\partial} h - \widetilde{\partial}\phi$ and Lemma 2.12.1, we have (2.12.27). $\square$

For the next estimate, we write $\widetilde{\mathscr{E}}^s = \mathcal{W}^s + \mathcal{E}^s + \varepsilon^2 \mathcal{E}^s_\varepsilon$ for the part of $\mathscr{E}$ that does not involve $\mathcal{A}$. We have:

Corollary 2.12.2. *For each $s \geq 0$, there is a continuous function $\mathcal{C}_s$ depending on M', L, δ^{-1}, T, $\mathcal{A}^{s-1}(0)$, $\mathcal{W}^{s-1}(0)$, $\sup_{0\leq t\leq T} \widetilde{\mathscr{E}}^{s-1}(t)$ so that if $V\in \mathcal{X}^{s+1}(T)$ satisfies (2.4.11), then with $\mathcal{A}_s$ as in (2.12.6):*

$$\mathcal{A}_s(t) \leq \mathcal{C}_s\big(\mathcal{A}_s(0) + \int_0^t (1+\tau)\widetilde{\mathscr{E}}^s(\tau)\, d\tau\big). \tag{2.12.33}$$

Proof. Using Lemmas 2.A.4 and 2.12.1, we have:

$$\begin{aligned} ||[J_\varepsilon, \operatorname{div}]V||^2_{H^{s-1}} + ||[J_\varepsilon, \operatorname{curl}]V||^2_{H^{s-1}} &\leq \varepsilon^2 C'(||\widetilde{x}||^2_{H^{s+1}} + ||V||^2_{H^{s+1}}) \\ &\leq C'(\mathcal{A}^s + (1+\delta^{-1})\widetilde{\mathscr{E}}^s), \end{aligned} \tag{2.12.34}$$

with $C' = C'(M', L, \delta^{-1}, \mathcal{A}^{s-1}, \widetilde{\mathscr{E}}^{s-1})$, and with $H^k = H^k(\Omega)$, noting that the highest-order term in the second inequality is multiplied by ε^2. Integrating (2.12.26) once in time, we have:

$$|| \operatorname{div} J_\varepsilon \partial_y x(t)||^2_{H^{s-1}(\Omega)} \leq || \operatorname{div} J_\varepsilon \partial_y x(0)||^2_{H^{s-1}(\Omega)} + \int_0^t ||D_t \operatorname{div} J_\varepsilon \partial_y x(\tau)||^2_{H^{s-1}(\Omega)}\, d\tau. \tag{2.12.35}$$

If $|I| = s-1$ then with R^I as defined in Lemma 2.12.3, then integrating (2.12.27) twice in time, we also have:

$$\begin{aligned} &||\partial_y^I \operatorname{curl} J_\varepsilon \partial_y x(t)||^2_{L^2} \\ &\quad\leq ||\partial_y^I \operatorname{curl} J_\varepsilon \partial_y x_0||^2_{L^2} + \int_0^t ||D_t \partial_y^I \operatorname{curl} \partial_y J_\varepsilon x_0 - R_0^I||^2_{L^2} + ||R^I(\tau)||^2_{L^2}\, d\tau \\ &\qquad + \int_0^t \int_0^\tau ||D_t^2 \partial_y^I \operatorname{curl} \partial_y J_\varepsilon x(\tau') - D_t R^I(\tau')||^2_{L^2}\, d\tau' d\tau, \end{aligned} \tag{2.12.36}$$

with $L^2 = L^2(\Omega)$ and $R_0^I = R^I|_{t=0}$. We have

$$||D_t \partial_y^I \operatorname{curl} \partial_y J_\varepsilon x(0) - R^I(0)||_{L^2(\Omega)} \leq \mathcal{C}_s''(M, L, \delta^{-1}, \mathcal{A}^{s-1}(0), \widetilde{\mathscr{E}}_0^{s-1})(\mathcal{A}^s(0) + \widetilde{\mathscr{E}}_0^s).$$

We now use the facts that div$V = -e'(h)D_th$, curl $D_tV = 0$, the estimates (2.12.26)- (2.12.27). Using (2.12.34) and Lemma 2.12.3 for R, we get:

$$\mathcal{A}^s(t) \leq \mathcal{A}^s(0) + \mathcal{C}_s'\Big(\int_0^t \mathcal{A}^s(\tau) + \widetilde{\mathscr{E}}^s(\tau)\,d\tau + \int_0^t\int_0^\tau \mathcal{A}^s(\tau') + \widetilde{\mathscr{E}}^s(\tau')\,d\tau' d\tau\Big) \tag{2.12.37}$$

$$\leq \mathcal{A}^s(0) + \mathcal{C}_s'\Big(\int_0^t (1+\tau)\mathcal{A}^s(\tau)\,d\tau + \int_0^t (1+\tau)\widetilde{\mathscr{E}}^s(\tau)\,d\tau\Big). \tag{2.12.38}$$

with $\mathcal{C}_s' = \mathcal{C}_s'(M', L, \delta^{-1}, \mathcal{A}^{s-1}(t), \widetilde{\mathscr{E}}^{s-1}(t), \mathcal{A}^{s-1}(0))$. We now assume that we have the estimate (2.12.33) for $s = 0, ..., m-1$. By the inductive assumption, (2.12.38) holds with $s = m$ and with $\mathcal{C}_m'$ replaced with $\mathcal{C}_m''$ depending on $M', L, \delta^{-1}, T, \mathcal{A}^{m-1}(0), \mathcal{W}^{m-1}(0), \sup_{0\leq t\leq T} \widetilde{\mathscr{E}}^{m-1}(t)$. Making this substitution into (2.12.38) with $s = m$ and letting $H(t)$ denote the right-hand side, we have that $H'(t) \leq \mathcal{C}_m''((1+t)\widetilde{\mathscr{E}}^m(t) + (1+t)H(t))$. Multiplying both sides by the integrating factor $e^{-(t+t^2/2)\mathcal{C}_m''}$ and integrating gives the result. □

Combining Lemmas 2.12.1, 2.12.2 and Corollary 2.12.2, we have:

Corollary 2.12.3. *With the same hypotheses as Lemmas 2.12.1-2.12.2, there are continuous functions $\mathcal{C}_s$ with $\mathcal{C}_s = \mathcal{C}_s\big(M, L, \delta^{-1}, T, \mathcal{A}^{s-1}(0), \mathcal{W}^{s-1}(0), \sup_{0\leq t\leq T} \mathscr{E}^{s-1}(t)\big)$ so that for $1 \leq s \leq r-1$: if $0 \leq t \leq T$:*

$$||\widetilde{x}(t)||^2_{H^{s+1}(\Omega)} + ||\widetilde{\partial} h(t)||^2_s + ||V(t)||^2_{H^{(s,1/2)}(\Omega)} + \varepsilon^2||V(t)||^2_{H^{s+1}(\Omega)} \leq \mathcal{C}_s \sup_{0\leq\tau\leq T} \mathscr{E}^s(\tau), \tag{2.12.39}$$

Proof of Theorem 2.12.1. We will prove that:

$$\mathscr{E}^s(t) \leq \mathcal{F}_s'\big(M', L, \delta^{-1}, T, \sup_{0\leq t\leq T}\mathscr{E}^{s-1}(t)\big)\Big(\mathscr{E}_0^s + \int_0^t (1+\tau)^2\mathscr{E}^s(\tau)\,d\tau\Big), \qquad 0 \leq t \leq T, \tag{2.12.40}$$

for a continuous function $\mathcal{F}_s'$. If the estimate (2.12.8) holds for $s = 0, ..., m-1$ then (2.12.40) implies:

$$\mathscr{E}^m(t) \le \mathcal{F}_m''(M', L, \delta^{-1}, T, \mathscr{E}_0^{m-1})\Big(\mathscr{E}_0^m + \int_0^t (1+\tau)^2 \mathscr{E}^s(\tau)\, d\tau\Big), \qquad 0 \le t \le T. \tag{2.12.41}$$

Letting $H_m(t)$ denote the right-hand side of this expression then $dH_m/dt \le (1+t)^2 \mathcal{F}_m'' H_m$ and so multiplying by $e^{-((1+t)^3/3-1)\mathcal{F}_m''}$ and integrating shows that (2.12.8) holds for $s = m$ as well.

By Lemma 2.12.2 and Corollary 2.12.3, we have shown that $\mathcal{A}^s, \mathcal{W}^s$ are bounded by the right-hand side of (2.12.40) and so it just remains to prove that $\mathcal{E}^s + \varepsilon^2 \mathcal{E}_\varepsilon^s$ is bounded by the right-hand side of (2.12.40). We will prove that, with $\mathcal{C}_s' = \mathcal{C}_s'\big(M', L, \delta^{-1}, T, \sup_{0\le t\le T} \mathscr{E}^{s-1}(t)\big)$:

$$\begin{aligned}&\frac{d}{dt}(\mathcal{E}^s(t) + \varepsilon^2 \mathcal{E}_\varepsilon^s(t))\\ &\le \mathcal{C}_s'\Big(\mathcal{E}^s(t) + \varepsilon^2\mathcal{E}_\varepsilon^s(t) + \mathcal{A}^s(t) + \mathcal{W}^s(t) + \int_0^t (1+\tau)\big(\mathcal{E}^s(\tau) + \varepsilon^2\mathcal{E}_\varepsilon^s(\tau) + \mathcal{A}^s(\tau) + \mathcal{W}^s(\tau)\big) d\tau\Big).\end{aligned} \tag{2.12.42}$$

Multiplying both sides of (2.12.42) by the integrating factor $e^{-t\mathcal{C}_s'}$ gives:

$$\mathcal{E}_\varepsilon^s(t) + \varepsilon^2 \mathcal{E}_\varepsilon^s(t) \le \mathcal{F}_s'\Big(\mathcal{E}^s(0) + \varepsilon^2 \mathcal{E}_\varepsilon^s(0) + \int_0^t (1+\tau)^2 \mathscr{E}^s(\tau)\, d\tau\Big), \tag{2.12.43}$$

with $\mathcal{F}_s' = \mathcal{F}_s'\big(M', L, \delta^{-1}, T, \sup_{0\le t\le T} \mathscr{E}^{s-1}(t)\big)$. Together with the estimates for $\mathcal{A}, \mathcal{W}$, this proves (2.12.40).

We start by controlling the time derivative of K^s. By (2.4.11), $\operatorname{curl} D_t V = 0$ and so $D_t \operatorname{curl} V = -(\widetilde{\partial} S_\varepsilon V)(\widetilde{\partial} V)$ by (2.D.2). Using also (2.A.10) to control $[\langle\partial_\theta\rangle_\mu^{1/2}, \operatorname{curl}]V$ and the product estimate (2.A.45), it follows that:

$$||D_t \operatorname{curl}\langle\partial_\theta\rangle_\mu^{1/2} V||_{H^{s-1}(\Omega)} \le C(M, ||\widetilde{x}||_{H^s(\Omega)}, ||V||_{H^s(\Omega)}) ||\langle\partial_\theta\rangle_\mu^{1/2} V||_{H^s(\Omega)}, \tag{2.12.44}$$

and so using Corollary 2.12.3 and induction, this implies that $dK^s/dt \le C(M)\mathscr{E}^s$. To control

K^s_ε, the same argument allows us to control the curl term and to control the divergence term, we write:

$$D_t \,\mathrm{div}\, \partial V = -\partial(\widetilde{\Delta} h + \widetilde{\Delta}\phi) - [D_t, \mathrm{div}]\partial V + [\mathrm{div}, \partial](\widetilde{\partial} h + \widetilde{\partial}\phi). \tag{2.12.45}$$

Since $[D_t, \mathrm{div}]\partial V = -(\widetilde{\partial}_i S_\varepsilon V^k)\widetilde{\partial}_k \partial V^i$ and $[\mathrm{div}, \partial](\widetilde{\partial} h + \widetilde{\partial}\phi) = -(\widetilde{\partial}_i \partial \widetilde{x}^k)\widetilde{\partial}_k(\widetilde{\partial}_i h + \widetilde{\partial}_i \phi)$, after using the product rule (2.A.45), the wave equation (2.4.7) along with Lemmas 2.D.9,2.D.7, the definition $\widetilde{\Delta}\phi = 4\pi\rho$, and Corollary 2.12.3, we can bound $\varepsilon^2 d\, ||\,\mathrm{div}\,\partial V||^2_{H^{s-1}(\Omega)}/dt$ by the right-hand side of (2.12.42).

It remains to prove that for $\mu = 0, ..., N$ and $|I| = s-1$, $d(\mathcal{E}^{I,\mu} + \varepsilon^2 \mathcal{E}^I_\varepsilon)/dt$ is bounded by the right-hand side of (2.12.42), and for this we use the energy identity (2.10.16) and an approximation argument. We could approximate $V, h, \widetilde{x}$ by smooth functions but since $\widetilde{x}$ is smooth in tangential directions and we only apply tangential derivatives, it will suffice to just approximate V, h. We start by noting that under our hypotheses, $V, D_t V, D_t h, \widetilde{\partial} h \in H^r(\Omega)$. Indeed, $D_t V = -\widetilde{\partial} h - \widetilde{\partial}\phi$ and so by Corollary 2.12.3 and Theorem 2.7.1, we have $D_t V, \widetilde{\partial} h \in H^r(\Omega)$. To see that $D_t h \in H^r(\Omega)$, we write $\partial_a D_t h = A^i{}_a \widetilde{\partial}_i D_t h = A^i{}_a D_t \widetilde{\partial}_i h + A^i{}_a \widetilde{\partial}_i S_\varepsilon V^k \widetilde{\partial}_k h$. The $H^{r-1}(\Omega)$ norm of the first term here is bounded by $C(M')||\widetilde{x}||_{H^{r+1}(\Omega)}||\widetilde{\partial} h||_r$ using (2.D.2) and the fact that $H^{r+1}(\Omega)$ is an algebra. The second term here is bounded by $C(M)||V||_{H^r(\Omega)}||\widetilde{\partial} h||_r$ for the same reason. By (2.12.1) we control $||\widetilde{\partial} h||_r$ and thus $||D_t h||_{H^r(\Omega)}$.

Therefore, there is a sequence of smooth vector fields $V_{(n)}$ and a sequence of smooth functions $h_{(n)}$ with $h_{(n)}|_{\partial\Omega} = 0$ so that

$$(V_{(n)}(t,\cdot), D_t V_{(n)}(t,\cdot), D_t h_{(n)}(t,\cdot), \widetilde{\partial} h_{(n)}(t,\cdot)), \to (V(t,\cdot), D_t V(t,\cdot), D_t h(t,\cdot), \widetilde{\partial} h(t,\cdot)) \tag{2.12.46}$$

in $H^r(\Omega)$. We claim that for all I, J with $|I| = s \le r-1$, $|J| = s-1$ and all $\mu = 0, ..., N$, we

have:

$$\widetilde{\partial}\langle\partial_\theta\rangle_\mu^{1/2}T^I h_{(n)} \to \widetilde{\partial}\langle\partial_\theta\rangle_\mu^{1/2}T^I h, \quad \langle\partial_\theta\rangle_\mu^{1/2}\,\mathrm{div}\,T^J\langle\partial_\theta\rangle_\mu^{1/2}V_{(n)} \to \langle\partial_\theta\rangle_\mu^{1/2}\,\mathrm{div}\,T^J\langle\partial_\theta\rangle_\mu^{1/2}V, \tag{2.12.47}$$

with the convergence in $L^2(\Omega)$. These claims follow after writing $\langle\partial_\theta\rangle_\mu^{1/2}\mathrm{div}\,T^J\langle\partial_\theta\rangle_\mu^{1/2}V_{(n)} = \langle\partial_\theta\rangle_\mu^{1/2}\langle\partial_\theta\rangle_\mu^{1/2}T^J\mathrm{div}\,V_{(n)} + \langle\partial_\theta\rangle_\mu^{1/2}[T^J\langle\partial_\theta\rangle_\mu^{1/2},\mathrm{div}]V_{(n)}$ and $\widetilde{\partial}\langle\partial_\theta\rangle_\mu^{1/2}T^I h_{(n)} = \langle\partial_\theta\rangle_\mu^{1/2}T^I\widetilde{\partial}h_{(n)} + [\langle\partial_\theta\rangle_\mu^{1/2}T^i,\widetilde{\partial}]h_{(n)}$. In each of these expressions, the first term converges in $L^2(\Omega)$. The commutator terms involve tangential derivatives of $\widetilde{x}$ to highest order and lower-order norms of $V_{(n)}$ and so these converge as well. By the continuity of the trace map:

$$T^I\langle\partial_\theta\rangle_\mu^{1/2}V_{(n)} \to T^I\langle\partial_\theta\rangle_\mu^{1/2}V, \quad D_tT^I\langle\partial_\theta\rangle_\mu^{1/2}V_{(n)} \to D_tT^I\langle\partial_\theta\rangle_\mu^{1/2}V, \tag{2.12.48}$$

with the convergence in $L^2(\partial\Omega)$.

We now apply Proposition 2.10.16 with $\alpha = V_{(n)}, q = h_{(n)}, \chi = \widetilde{\partial}S_\varepsilon V$. For sufficiently large n, the assumptions (2.10.8)-(2.10.9) hold with $K = 2\delta + 2M' + 2L$. With $\mathcal{E}_n^I, \mathcal{E}_{n,\varepsilon}^I$ defined by (2.10.10)-(2.10.11) with $\alpha = V_{(n)}$, $q = h_{(n)}$ let $\mathcal{R}_n + \varepsilon\mathcal{R}_{n,\varepsilon} = \sum_{\mu=0}^N\sum_{|I|\le s}\mathcal{R}_n^{I,\mu} + \varepsilon\mathcal{R}_{n,\varepsilon}^{I,\mu}$, where $\mathcal{R}_{n,\varepsilon}^{I,\mu} = ||\gamma\cdot\langle\partial_\theta\rangle_\mu^{1/2}D_tT^IV_{(n)} - (\langle\partial_\theta\rangle_\mu^{1/2}T^I\gamma)\cdot\widetilde{\partial}h_{(n)}||_{L^2(\partial\Omega)}$ and

$$\begin{aligned}\mathcal{R}_n^{I,\mu} &= ||D_t\langle\partial_\theta\rangle_\mu^{1/2}T^IV_{(n)} - \widetilde{\partial}\big((T^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j)(\widetilde{\partial}_jh_{(n)}) + T^I\langle\partial_\theta\rangle_\mu^{1/2}h_{(n)}\big)||_{L^2(\Omega)}\\ &\quad + ||\langle\partial_\theta\rangle_\mu^{1/2}\big(e'(h)D_tT^Jh_{(n)} - (\widetilde{\partial}_iT^I\langle\partial_\theta\rangle_\mu^{1/2}\widetilde{x}^j)(\widetilde{\partial}_jS_\varepsilon V^i) + \mathrm{div}\,T^J\langle\partial_\theta\rangle_\mu^{1/2}V_{(n)}\big)||_{L^2(\Omega)}\\ &\quad + ||D_tT^I\langle\partial_\theta\rangle_\mu^{1/2}x - T^I\langle\partial_\theta\rangle_\mu^{1/2}V_{(n)}||_{L^2(\partial\Omega)},\end{aligned} \tag{2.12.49}$$

and $T^I = ST^J$ for $S \in \mathcal{T}, |J| = s-1$. The energy inequality (2.10.16) then gives, with $C_0 =$

$C_0(M', \delta^{-1})$,

$$\frac{d}{dt}\mathcal{E}_n^I \leq \sqrt{\mathcal{E}_n^s} C_0 \Big(\mathcal{R}_n + ||V_{(n)}||_{H^{(s,1/2)}(\Omega)} + ||J_\varepsilon x||_{H^s(\Omega)} + ||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}$$

$$+ ||h_{(n)}||_{H^{s+1}(\Omega)} + ||D_t h_{(n)}||_{H^s(\Omega)} \Big) + C_s \mathcal{E}_n^s + C_s \varepsilon ||V_{(n)}||_{H^{s+1/2}(\partial\Omega)} ||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}, \quad (2.12.50)$$

$$\frac{d}{dt}\mathcal{E}_{n,\varepsilon}^I \leq \sqrt{\mathcal{E}_{n,\varepsilon}^s} (\mathcal{R}_{n,\varepsilon} + \varepsilon^{-1} ||J_\varepsilon x||_{H^{s+1/2}(\partial\Omega)}). \quad (2.12.51)$$

By the above, using Lemma 2.11.1 to control $\lim_{n\to\infty} \mathcal{R}_n$ and Corollary 2.12.3 again, this implies (2.12.42). □

Before proving Corollary 2.12.1, we prove the following simple lemma:

Lemma 2.12.4. *Fix $r \geq 6$, and write $V_k^\varepsilon = D_t^k V|_{t=0}, h_k^\varepsilon = D_t^k h|_{t=0}$. Suppose that the bound (2.12.11) holds for x_0, V_k^ε and h_k^ε. There is a continuous function $\mathcal{T}_r = \mathcal{T}_r(M_0', L_0, \delta_0^{-1}, \mathscr{E}_0^r)$ so that if $T \leq \mathcal{T}_r$, and $V \in \mathcal{X}^{r+1}(T)$ satisfies the smoothed-out Euler equations (2.4.11), then for $0 \leq t \leq T$:*

$$||\partial\widetilde{x}(t,\cdot)/\partial y||_{L^\infty} + ||\partial y(t,\cdot)/\partial\widetilde{x}||_{L^\infty} +$$

$$\textstyle\sum_{|I|+k\leq 3} ||\partial_y^I D_t^k V(t,\cdot)||_{L^\infty} + ||\widetilde{x}(t,\cdot)||_{H^6(\Omega)} \leq 4M_0', \quad (2.12.52)$$

$$\textstyle\sum_{|I|+k\leq 3} ||\partial_y^I \widetilde{\partial} h(t,\cdot)||_{L^\infty} + ||D_t^k h(t,\cdot)||_{L^\infty} \leq 2L_0, \quad (2.12.53)$$

$$-\widetilde{\partial}_N h(t,\cdot)|_{\partial\Omega} \geq \delta_0/2. \quad (2.12.54)$$

Proof. Let $M_1(t) = ||\partial_y \widetilde{x}(t)||_{L^\infty(\Omega)} + \sum_{|I|+k\leq 3} ||\partial_y^I D_t^k V(t)||_{L^\infty(\Omega)}$ and $M_2(t) = ||\partial y(t)/\partial\widetilde{x}||_{L^\infty(\Omega)}$. Further, let $L(t) = \sum_{|I|+k\leq 3} ||\partial_y^I D_t^k \widetilde{\partial} h(t)||_{L^\infty(\Omega)} + ||D_t^k h(t)||_{L^\infty(\Omega)}$ and $\nu(t) = ||(-\widetilde{\partial}_N h(t))^{-1}||_{L^\infty(\partial\Omega)}$. Note that by the definition of $\widetilde{x}$ in (2.4.1) and the definition of V_k^ε, we have $M_1(0) + M_2(0) \leq M_0'$

and $L(0)=L_0$ and $\nu(0)\leq\delta_0^{-1}$. By Sobolev embedding, the fundamental theorem of calculus, and the fact that the operator J_ε is bounded:

$$M_1(t) \leq M_0' + C_1\Big(\int_0^t ||V(\tau)||_6\, d\tau\Big), \tag{2.12.55}$$

$$L(t) \leq L_2 + C_2\Big(\int_0^t ||D_t h(\tau)||_6 + ||\widetilde{\partial} h(\tau)||_6\, d\tau\Big). \tag{2.12.56}$$

Using the trace inequality (2.A.41), we also have:

$$\nu(t) \leq \delta_0^{-1} + \int_0^t \nu(\tau)^2 ||D_t\widetilde{\partial} h(\tau,\cdot)||_{L^\infty(\partial\Omega)}\, d\tau \leq \delta_0^{-1} + C_3\int_0^t \nu(\tau)^2||\widetilde{\partial} h(\tau)||_4\, d\tau. \tag{2.12.57}$$

Finally, integrating in time, using (2.D.2) and Sobolev embedding, we have:

$$M_2(t) \leq M_0' + C_3\int_0^t M_2(\tau)^2||V(\tau)||_4\, d\tau. \tag{2.12.58}$$

If $V \in \mathcal{X}^{r+1}(T_1)$ solves the smoothed Euler equations (2.4.11) for some $T_1 > 0$ then Corollary 2.12.3 combined with Theorem 2.12.1 gives a continuous function $\mathcal{F}_r'$ so that:

$$||V(\tau)||_6^2 + ||D_t h(\tau)||_6^2 + ||\widetilde{\partial} h(\tau)||_4^2 \leq \mathcal{F}_r'(M', L, \mathscr{E}_0^{r-1})\mathscr{E}_0^r, \qquad 0\leq\tau\leq T_1. \tag{2.12.59}$$

Here, the constants C_1, C_2, C_3 depend only on Ω. Set $\mathcal{F}_r''=C_0\mathcal{F}_r'(4M_0', 2L_0, (2\delta_0)^{-1}, \mathscr{E}_0^{r-1})(\mathscr{E}_0^r + (2\delta_0)^{-2})$ with $C_0 = C_1+C_2+C_3$, and define $\mathcal{T}_r = \min(M_0', 1/M_0', L_0, \delta_0)(16\mathcal{F}_r'')^{-1}$. Take $T\leq \min(\mathcal{T}, T_1)$ and consider the set:

$$S = \{0\leq t\leq T : M_1(t) + M_2(t) \leq 4M_0', L(t)\leq 2L_0, \nu(t)\leq 2\delta_0^{-1}\}. \tag{2.12.60}$$

Then S is nonempty, since it contains $t=0$, and it is connected and closed by continuity of the functions $M_1(t), M_2(t), L(t), \nu(t)$. If $t \in S$ then the assumption $T \leq \mathcal{T}_r$ and (2.12.56) imply:

$$M_1(t) \leq M_0' + T\mathcal{F}_r'' \leq M_0' + M_0'(16\mathcal{F}_r'')^{-1}\mathcal{F}_r'', \tag{2.12.61}$$

$$L(t) \leq L_0 + T\mathcal{F}_r'' \leq L_0 + L_0(16\mathcal{F}_r'')^{-1}\mathcal{F}_r'', \tag{2.12.62}$$

and similarly

$$M_2(t) \leq M_0' + (4M_0')^2(16M_0'\mathcal{F}_r'')^{-1}\mathcal{F}_r'', \qquad \nu(t) \leq \delta_0^{-1} + 2\delta_0^{-2}\delta_0(16\mathcal{F}_r'')^{-1}\mathcal{F}_r''. \tag{2.12.63}$$

In particular $M_1(t)+M_2(t) \leq 3M_0'$, $L(t) \leq 3L_0/2$ and $\nu(t) \leq 3\delta_0^{-1}/2$. Hence S is also open so $S = \{0 \leq t \leq T\}$. □

Proof of Corollary 2.12.1. Let $\mathcal{T}_r$ be as in Lemma 2.12.4 and with $\mathcal{F}_r$ as in Theorem 2.12.1, define:

$$\mathscr{T}_r = \mathcal{T}_r, \qquad \mathscr{C}_r = \mathcal{F}_r(4M_0', 2L_0, (2\delta_0)^{-1}, \mathscr{E}_0^{r-1}). \tag{2.12.64}$$

By (2.12.8) and Lemma 2.12.4, this proves (2.12.12). The estimate (2.12.13) follows from (2.12.12) and Corollary 2.12.3. □

2.A Fractional tangential derivatives and tangental smoothing

There is a family of open sets V_μ, $\mu = 1, \dots, N$ that cover $\partial\Omega$ and onto diffeomorphisms $\Phi_\mu : (-1,1)^2 \to V_\mu$. We fix a collection of cutoff functions $\chi_\mu : \partial\Omega \to \mathbb{R}$ so that χ_μ^2 form a partition of unity and $\operatorname{supp}\chi_\mu \subset V_\mu$, as well as two other families of cutoff functions such that $\widetilde{\chi}_\mu \equiv 1$ on $\operatorname{supp}\chi_\mu$, $\overline{\chi}_\mu \equiv 1$ on $\operatorname{supp}\widetilde{\chi}_\mu$ and $\operatorname{supp}\overline{\chi}_\mu \subset V_\mu$. Recalling that Ω is the unit ball, we set $W_\mu = \{r\omega, r \in (1/2, 1], \omega \in V_\mu\}$ for $\mu = 1, \dots, N$ and let W_0 be the ball of radius $3/4$ so that $\{W_\mu\}_{\mu=0}^N$ covers Ω. Writing $\Psi_\mu(z, z_3) = z_3\Phi_\mu(z)$, Ψ_μ is a diffeomorphism from $(-1,1)^2 \times (1/2,1]$ to W_μ. Let $\zeta : [0,1] \to \mathbb{R}$ be a bump function so that $\zeta(r) = 1$ when

$1/2 \leq r \leq 1$ and $\zeta(r) = 0$ when $r < 1/4$. We extend the above cutoffs to Ω by setting $\chi_\mu(y) = \chi_\mu(y/|y|)\zeta(|y|)$ for $\mu = 1, ..., N$ and $\chi_0 = 1 - \zeta$, and we similarly extend $\widetilde{\chi}_\mu$ and $\overline{\chi}_\mu$. We abuse notation by writing χ_μ also for the function $\chi_\mu \circ \Psi_\mu$

2.A.1 Fractional derivatives

For a function $F : \mathbb{R}^2 \to \mathbb{R}$, we set:

$$\langle\partial_\theta\rangle^{1/2} F(z) = \int_{\mathbb{R}^2} \langle\xi\rangle^{1/2} \hat{F}(\xi) e^{iz\cdot\xi}\, d\xi, \quad \text{where } \hat{F}(\xi) = \int_{\mathbb{R}^2} e^{-iz\cdot\xi} F(z)\, dz. \tag{2.A.1}$$

Given a function $f : \Omega \to \mathbb{R}$, we define $\langle\partial_\theta\rangle_\mu^{1/2} f : \Omega \to \mathbb{R}$ for $\mu = 1, ..., N$ by:

$$\langle\partial_\theta\rangle_\mu^{1/2} f = \widetilde{\chi}_\mu (\langle\partial_\theta\rangle^{1/2} f_\mu) \circ \Psi_\mu^{-1}, \quad \text{where} \quad f_\mu = (\chi_\mu f) \circ \Psi_\mu : \mathbb{R}^2 \to \mathbb{R}. \tag{2.A.2}$$

With the cutoff function ζ defined above, we let $\mathcal{T}$ denote the following family of vector fields, which span the tangent space to the boundary and in the interior span the full tangent space:

$$\zeta(y)(y^a \partial_{y^b} - y^b \partial_{y^a}), \quad (1 - \zeta(y))\partial_{y^a}, \quad a, b = 1, 2, 3. \tag{2.A.3}$$

We work in terms of the following Sobolev norms, for $s \in \mathbb{R}$:

$$||f||^2_{H^s(\partial\Omega)} = \sum_{\mu=1}^N ||\langle\partial_\theta\rangle^s f_\mu||^2_{L^2(\mathbb{R}^2)} = \sum_{\mu=1}^N \int_{\mathbb{R}^2} |\langle\xi\rangle^s \hat{f}_\mu(\xi)|^2\, d\xi, \tag{2.A.4}$$

and if $s \in \mathbb{R}, k \in \mathbb{N}$ we set:

$$||f||^2_{H^{(k,s)}(\Omega)} = \sum_{|I| \leq k} \int_0^1 ||\partial_y^I (\zeta f)(r, \cdot)||^2_{H^s(\partial\Omega)}\, r^2 dr + ||(1 - \zeta) f||^2_{H^{k+s}(\Omega)}, \tag{2.A.5}$$

where for non-integer s, $H^{k+s}(\Omega)$ is defined in the usual way by taking the Fourier transform in all variables. We collect here the basic properties of the operators $\langle\partial_\theta\rangle_\mu^{1/2}$ and the norms $H^s(\partial\Omega), H^{(k,s)}(\Omega)$:

Lemma 2.A.1. *If $T \in \mathcal{T}$, then:*

$$\Big| \int_{\partial\Omega} f T g \, dS(y) \Big| \leq C ||f||_{H^{1/2}(\partial\Omega)} ||g||_{H^{1/2}(\partial\Omega)}, \tag{2.A.6}$$

$$\Big| \int_{\Omega} f T g \, dy \Big| \leq C ||f||_{H^{(0,1/2)}(\Omega)} ||g||_{H^{(0,1/2)}(\Omega)}. \tag{2.A.7}$$

In addition, with $\Sigma = \partial\Omega$ or Ω,

$$||\langle \partial_\theta \rangle_\mu^{1/2} (fg) - f \langle \partial_\theta \rangle_\mu^{1/2} g||_{L^2(\Sigma)} \leq C ||f||_{H^2(\Sigma)} ||g||_{L^2(\Sigma)}, \tag{2.A.8}$$

and, with notation as in (2.3.17) *and $T^I \in \mathcal{D}^k$ or $\mathcal{T}^k$:*

$$||\langle \partial_\theta \rangle_\mu^{1/2} (T^I f) - T^I \langle \partial_\theta \rangle_\mu^{1/2} f||_{L^2(\Sigma)} \leq C ||f||_{H^k(\Sigma)}. \tag{2.A.9}$$

In particular, if $||\widetilde{x}||_{H^3(\Omega)} \leq M$ then:

$$||\langle \partial_\theta \rangle_\mu^{1/2} \widetilde{\partial} f - \widetilde{\partial} \langle \partial_\theta \rangle_\mu^{1/2} f||_{L^2(\Sigma)} \leq C(M) ||f||_{H^1(\Sigma)}. \tag{2.A.10}$$

These estimates all rely on the following "Leibniz rule". This lemma and its proof can be found in [2].

Lemma 2.A.2. *If $F, G : \mathbb{R}^2 \to \mathbb{R}$ have compact support, then:*

$$||\langle \partial_\theta \rangle^{1/2} (FG) - F \langle \partial_\theta \rangle^{1/2} G||_{L^2(\mathbb{R}^2)} \leq C ||F||_{H^2(\mathbb{R}^2)} ||G||_{L^2(\mathbb{R}^2)}. \tag{2.A.11}$$

Proof. By the elementary estimate $|\langle \xi \rangle^{1/2} - \langle \xi - \eta \rangle^{1/2}| \leq C \langle \eta \rangle^{1/2}$, we have:

$$\begin{aligned} |\langle \xi \rangle^{1/2} \widehat{FG}(\xi) - (F \widehat{\langle \partial_\theta \rangle^{1/2}} G)(\xi)|^2 &\lesssim \Big(\int_{\mathbb{R}^2} \langle \eta \rangle^{1/2} |\hat{F}(\eta)| |\hat{G}(\xi - \eta)| \, d\eta \Big)^2 \\ &\lesssim \int_{\mathbb{R}^2} \langle \eta \rangle^4 |\hat{F}(\eta)|^2 \, d\eta \int_{\mathbb{R}^2} \langle \eta \rangle^{-3} |\hat{G}(\xi - \eta)|^2 \, d\eta. \end{aligned} \tag{2.A.12}$$

Integrating in ξ, changing variables, and using the fact that $\int_{\mathbb{R}^2} \langle \xi - \eta \rangle^{-3} \, d\xi \leq C$, we have:

$$||\langle \xi \rangle \widehat{FG} - (\widehat{F \langle \partial_\theta \rangle^{1/2} G})||^2_{L^2(\mathbb{R}^2)} \leq C ||F||^2_{H^2(R)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle \xi - \eta \rangle^{-3} |\hat{G}(\eta)|^2 \, d\eta \, d\xi$$

$$\leq C ||F||_{H^2(R)} ||G||_{L^2(R)}. \quad (2.A.13)$$

The result now follows from Plancherel's theorem. □

Proof of Lemma 2.A.1. Since $\sum \chi_\mu^2 = 1$, we have:

$$\int_{\partial\Omega} f T g \, dS(y) = \sum_{\mu=1}^N \int_{\partial\Omega} (\chi_\mu f)(\chi_\mu T g) \, dS(y)$$

$$= \sum_{\mu=1}^N \int_{\partial\Omega} \chi_\mu f T(\chi_\mu g) \, dS(y) - \int_{\partial\Omega} \chi_\mu f g T \chi_\mu \, dS(y). \quad (2.A.14)$$

The second term is bounded by $C||f||_{L^2(\partial\Omega)} ||g||_{L^2(\partial\Omega)}$. To deal with the first term, we use (2.A.2) and write:

$$\int_{\partial\Omega} \chi_\mu f T(\chi_\mu g) \, dS(y) = \int_{\mathbb{R}^2} f_\mu T^\alpha \partial_{z^\alpha} g_\mu |\det \Phi'_\mu| \, dz, \quad \text{where} \quad T = T^\alpha \partial_{z^\alpha}. \quad (2.A.15)$$

With $F^\alpha = f_\mu T^\alpha |\det \Phi'_\mu|$ and $G = g_\mu$, by Plancherel's theorem we have:

$$\int_{\mathbb{R}^2} F^\alpha(z) \partial_{z^\alpha} G(z) dz = \int_{\mathbb{R}^2} \hat{F}^\alpha(\xi) i \xi_\alpha \hat{G}(\xi) \, d\xi \leq ||\langle \xi \rangle^{1/2} \hat{F}||_{L^2(\mathbb{R}^2)} ||\langle \xi \rangle^{1/2} \hat{G}||_{L^2(\mathbb{R}^2)}. \quad (2.A.16)$$

By (2.A.11) and (2.A.4), this is bounded by $(||\langle \partial_\theta \rangle^{1/2} f_\mu||_{L^2(R)} + ||f||_{L^2(\partial\Omega)}) ||g||_{H^{1/2}(\partial\Omega)}$. The case $\Sigma = \Omega$ is similar.

We now prove (2.A.8). Writing $\overline{f}_\mu = \overline{\chi}_\mu f \circ \Psi_\mu$, where $\overline{\chi}_\mu \equiv 1$ in the support of $\widetilde{\chi}_\mu$ in

(2.A.2), we have:

$$||\langle\partial_\theta\rangle_\mu^{1/2}(fg) - f\langle\partial_\theta\rangle_\mu^{1/2} g||_{L^2(\partial\Omega)} \lesssim ||\langle\partial_\theta\rangle^{1/2}(\overline{f}_\mu g_\mu) - \overline{f}_\mu\langle\partial_\theta\rangle^{1/2} g_\mu||_{L^2(\mathbb{R}^2)}$$

$$\lesssim \|\overline{f}_\mu\|_{H^2(\mathbb{R}^2)}\|g_\mu\|_{L^2(\mathbb{R}^2)} \lesssim ||f||_{H^2(\partial\Omega)}||g||_{L^2(\partial\Omega)}, \quad (2.A.17)$$

by (2.A.11), which gives (2.A.8) for $\Sigma = \partial\Omega$. The case $\Sigma = \Omega$ follows from the case $\Sigma = \partial\Omega$ by the definition (2.A.4).

We now prove (2.A.9). We first prove the case $k = 1$ with $T \in \mathcal{T}$ and $\Sigma = \partial\Omega$. Since $\partial_{z^\alpha}\langle\partial_\theta\rangle^{1/2} = \langle\partial_\theta\rangle^{1/2}\partial_{z^\alpha}$:

$$T^\alpha\partial_{z^\alpha}\langle\partial_\theta\rangle^{1/2} f_\mu - \langle\partial_\theta\rangle^{1/2}(Tf)_\mu = T^\alpha\langle\partial_\theta\rangle^{1/2}(\partial_\alpha f_\mu) - \langle\partial_\theta\rangle^{1/2}(T^\alpha\partial_\alpha f_\mu) + \langle\partial_\theta\rangle^{1/2}((T^\alpha\partial_\alpha\chi_\mu)f), \quad (2.A.18)$$

Applying (2.A.11), the L^2 norm of the right-hand side is bounded by $C||f||_{H^1(\partial\Omega)}$. The commutator of $T\langle\partial_\theta\rangle_\mu^{1/2} f - \langle\partial_\theta\rangle_\mu^{1/2} Tf$ just contribute an additional term $(T\widetilde{\chi}_\mu)\langle\partial_\theta\rangle^{1/2} f_\mu$ compared to (2.A.18) and (2.A.9) follows.

To prove (2.A.9) when $T = \partial_{y^a}$ for some $a = 1, 2, 3$ and $\Sigma = \partial\Omega$, close to the boundary we write $\partial_{y^a} = \sum_{T\in\mathcal{T}} c_a^T(y)T + c(y)\partial_r$ for some smooth functions c_a^T and c. By what we have just proven and (2.A.8), it is enough to prove the estimate with T replaced by ∂_r. This follows from the definition after noting that close to the boundary, the cutoff functions $\widetilde{\chi}_\mu, \chi_\mu$ are independent of r. The case $|I| \geq 2$ follows similarly. □

The operators $\langle\partial_\theta\rangle_\mu^{1/2}$ can be used to control fractional Sobolev norms:

Lemma 2.A.3. *We have*

$$||(1 - \widetilde{\chi}_\mu)\langle\partial_\theta\rangle^{1/2} f_\mu||_{L^2(\mathbb{R}^2)} \lesssim ||f_\mu||_{L^2(\mathbb{R}^2)}. \quad (2.A.19)$$

Moreover

$$\sum_{\mu=1}^{N} ||\langle\partial_\theta\rangle_\mu^{1/2} f||_{L^2(\partial\Omega)} + ||f||_{L^2(\partial\Omega)} \sim ||f||_{H^{1/2}(\partial\Omega)}, \tag{2.A.20}$$

The same estimate holds with $\partial\Omega$ replaced by Ω and $H^{1/2}(\partial\Omega)$ replaced with $H^{(0,1/2)}(\Omega)$.

Here, we are writing $A \sim B$ to mean that there are constants C_1, C_2 so that $C_1 A \leq B \leq C_2 B$.

Proof. Since $\widetilde{\chi}_\mu = 1$ on the support of χ_μ and hence on the support of f_μ it follows from (2.A.11) that

$$||(1-\widetilde{\chi}_\mu)\langle\partial_\theta\rangle^{1/2} f_\mu||_{L^2(R)}$$

$$= ||\langle\partial_\theta\rangle^{1/2}(\widetilde{\chi}_\mu f_\mu) - \widetilde{\chi}_\mu\langle\partial_\theta\rangle^{1/2} f_\mu||_{L^2(R)} \leq C||f_\mu||_{L^2(R)} \leq C||f||_{L^2(\partial\Omega)}. \qquad \square$$

2.A.2 Tangential smoothing

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be an even smooth function, supported in $R = (-1,1)^2$, with $\int_{\mathbb{R}^2} \varphi = 1$ and define the smoothing operator

$$T_\varepsilon f(z) = \int_{\mathbb{R}^2} \varphi_\varepsilon(z-z') f(z') dz', \qquad \text{where} \qquad \varphi_\varepsilon(z) = \varepsilon^{-2}\varphi(z/\varepsilon). \tag{2.A.21}$$

Because φ is even, T_ε is symmetric; for any functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ we have:

$$\int_{\mathbb{R}^2} T_\varepsilon f(z) g(z) dz = \int_{\mathbb{R}^2}\int_R \varphi_\varepsilon(z-z') f(z') g(z) dz'\, dz = \int_{\mathbb{R}^2} f(z) T_\varepsilon g(z) dz. \tag{2.A.22}$$

From the fact that $\|\partial^k \varphi_\varepsilon\|_{L^1} \lesssim \varepsilon^{-k}$ it follows that for $k \geq m$

$$\|T_\varepsilon f\|_{H^k} \lesssim \varepsilon^{m-k} \|f\|_{H^m}. \tag{2.A.23}$$

Furthermore, we have:

$$|T_\varepsilon(fg)(z) - f T_\varepsilon(g)(z)| \leq C\varepsilon ||f||_{C^1(R)} ||g||_{L^2(R)}, \tag{2.A.24}$$

which follows from the fact that $|z'| \leq \varepsilon$ in the support of φ_ε, after writing:

$$T_\varepsilon(fg)(z) - f(z)T_\varepsilon(g)(z) = \int_{\mathbb{R}^2} \varphi_\varepsilon(z')g(z-z')\big(f(z-z') - f(z)\big)dz'. \tag{2.A.25}$$

Moreover from using (2.A.40) and Minkowski's integral inequality in (2.A.25) with $g=1$ it follows that

$$\|T_\varepsilon f - f\|_{H^k} \lesssim \varepsilon \|f\|_{H^{k+1}}. \tag{2.A.26}$$

For a linear operator T' defined in coordinate charts we define a global operator T by

$$Tf = \sum T_\mu f, \quad \text{where} \quad T_\mu f = \chi_\mu\big(m_\mu^{-1}T'\big[m_\mu f_\mu\big]\big) \circ \Psi_\mu^{-1}, \qquad f_\mu = (\chi_\mu f) \circ \Psi_\mu, \tag{2.A.27}$$

where $m_\mu = r|\det \Phi_\mu'|^{1/2}$. Then T is symmetric with the measure dy if T' is with the measure dz is since $dS(\omega) = m_\mu^2 dz$. With notation as in (2.A.27), the smoothing operators we consider on Ω or $\partial\Omega$ are then defined by:

$$J_\varepsilon f = \textstyle\sum_{\mu=0}^N T_{\varepsilon,\mu} f, \qquad S_\varepsilon f = J_\varepsilon J_\varepsilon f = \textstyle\sum_{\mu,\nu=0}^N T_{\varepsilon,\nu} T_{\varepsilon,\mu} f. \tag{2.A.28}$$

Since T_ε is symmetric J_ε is as well, w.r.t. dy.

The smoothing operator has the following important properties:

Lemma 2.A.4. *If $f, g : \Omega \to \mathbb{R}$, then with $\Sigma = \partial\Omega$ or Ω*

$$||J_\varepsilon(fg) - fJ_\varepsilon(g)||_{L^2(\Sigma)} \leq C\varepsilon ||f||_{C^1(\Sigma)} ||g||_{L^2(\Sigma)}. \tag{2.A.29}$$

$$\Big| \int_{\partial\Omega} \big(f(S_\varepsilon g) - (J_\varepsilon f)(J_\varepsilon g)\big)\, \widetilde{\nu} dS(y) \Big| \leq C\varepsilon ||\widetilde{\nu}||_{C^1(\partial\Omega)} ||f||_{L^2(\Omega)} ||g||_{L^2(\partial\Omega)}, \tag{2.A.30}$$

$$\Big| \int_{\Omega} \big(f(S_\varepsilon g) - (J_\varepsilon f)(J_\varepsilon g)\big)\, \widetilde{\kappa} dy \Big| \leq C\varepsilon ||\widetilde{\kappa}||_{C^1(\Omega)} ||f||_{L^2(\Omega)} ||g||_{L^2(\Omega)}, \tag{2.A.31}$$

Further, if $T^I \in \mathcal{T}^k$ for $k \geq 0$:

$$||T^I J_\varepsilon f - J_\varepsilon(T^I f)||_{L^2(\Sigma)} \leq C||f||_{H^{k-1}(\Sigma)}, \tag{2.A.32}$$

$$||\langle\partial_\theta\rangle_\mu^{1/2} J_\varepsilon f - J_\varepsilon(\langle\partial_\theta\rangle_\mu^{1/2} f)||_{L^2(\Sigma)} \leq C||f||_{L^2(\Sigma)}. \tag{2.A.33}$$

Proof. The estimate (2.A.29) is a straightforward consequence of (2.A.24).

To prove (2.A.30) and (2.A.31) note first that J_ε is symmetric with respect to the measure $dS(y)$ since $m_\mu^2 dS(y) = dz$:

$$\int_{\partial\Omega} f J_\varepsilon g\, dS = \sum_\mu \int_{\partial\Omega} f \chi_\mu (m_\mu^{-1} T_\varepsilon[m_\mu g_\mu]) \circ \Psi_\mu^{-1} dS = \sum_\mu \int_R m_\mu f_\mu T_\varepsilon[m_\mu g_\mu] dz$$

$$= \sum_\mu \int_R T_\varepsilon[m_\mu f_\mu] m_\mu g_\mu dz = \int_{\partial\Omega} J_\varepsilon f\, g\, dS. \tag{2.A.34}$$

(2.A.30) follows from this applied to $\widetilde{\nu} f$ in place of f and then (2.A.29) with $\widetilde{\nu}$ in place of f and f in place of g.

Changing coordinates, using that $\partial_z(\varphi_\varepsilon * F) = \varphi_\varepsilon * (\partial_z F)$ for any function $F: (-1,1)^2 \to \mathbb{R}$ and using (2.A.29), a straightforward calculation as in the proof of Lemma 2.A.1 shows that $||T^I J_\varepsilon f - J_\varepsilon T^I f||_{L^2(\Sigma)} \lesssim ||f||_{H^{k-1}(\Sigma)}$.

(2.A.33) follows from that $[\langle\partial_\theta\rangle^{1/2}, T_\varepsilon] = 0$ and $\sum_\nu \chi_\nu^2 = 1$, after repeatedly using (2.A.8) and (2.A.29) in

$$\langle\partial_\theta\rangle_\mu^{1/2} J_\varepsilon f = \sum_\nu \widetilde{\chi}_\mu \Big(\langle\partial_\theta\rangle^{1/2} \big[\chi_\mu \chi_\nu m_\nu^{-1} T_\varepsilon\big[m_\nu f_\nu\big]\big]\Big) \circ \Psi_\nu^{-1}, \tag{2.A.35}$$

$$J_\varepsilon \langle\partial_\theta\rangle_\mu^{1/2} f = \sum_\nu \chi_\nu \Big(m_\nu^{-1} T_\varepsilon\big[\chi_\nu m_\nu \widetilde{\chi}_\mu \langle\partial_\theta\rangle^{1/2}[f_\mu]\big]\Big) \circ \Psi_\nu^{-1}. \qquad \square$$

2.A.3 Interpolation and Sobolev Inequalities

Here we collect some standard inequalities we will use.

We will use the Sobolev inequalities on both Ω and $\partial\Omega$. For any tensor field α on either $\Omega \cup \partial\Omega$ or $\partial\Omega$:

$$||\alpha||_{L^{3p/(3-kp)}(\Omega)} \leq C\sum_{|I|\leq k}||\partial_y^I \alpha||_{L^p(\Omega)}, \qquad 1 \leq p < 3/k, \tag{2.A.36}$$

$$||\alpha||_{L^\infty(\Omega)} \leq C\sum_{|I|\leq k}||\partial_y^I \alpha||_{L^p(\Omega)}, \qquad k > 3/p, \tag{2.A.37}$$

$$||\alpha||_{L^{2p/(2-kp)}(\partial\Omega)} \leq C\sum_{|I|\leq k}||\partial_y^I \alpha||_{L^p(\partial\Omega)}, \qquad 1 \leq p < 2/k, \tag{2.A.38}$$

$$||\alpha||_{L^\infty(\partial\Omega)} \leq \sum_{|I|\leq k}||\partial_y^I \alpha||_{L^p(\partial\Omega)}, \qquad k > 2/p. \tag{2.A.39}$$

By, e.g. the results in the appendix of [11], the constants above depend only on the injectivity radius of Ω.

We also have the following alternative characterization of the Sobolev spaces

$$\|D_c^h F\|_{L^2} \lesssim \|\partial_c F\|_{L^2} \lesssim \sup_h \|D_c^h F\|_{L^2}, \quad \text{where} \quad D_c^h F(z) = \big(F(z+he_c) - F(z)\big)/h, \tag{2.A.40}$$

denotes the difference quotient in the direction of a unit vector e_c, see [12].

We will also need the trace inequality (see, e.g. [13]):

$$||f||_{H^{s-1/2}(\partial\Omega)} \leq C||f||_{H^s(\Omega)}, \qquad s > 1/2. \tag{2.A.41}$$

We will only apply this when s is a positive integer and in that case the right-hand side is defined in the usual way and the left-hand side is defined by (2.3.11). We will use the following Sobolev inequalities.

Lemma 2.A.5. *If $s \geq 2$, then:*

$$||f||_{L^\infty(\Omega)} \leq C||f||_{H^s(\Omega)}. \tag{2.A.42}$$

Further, with notation as in Section 2.3.3, if $s \geq 2$ then:

$$||f||_{L^\infty(\Omega)} \leq C||\mathcal{T}^s f||_{H^1(\Omega)}. \tag{2.A.43}$$

If $k < 3/p$ and $1/q = 1/p - k/3$, then:

$$||f||_{L^p(\Omega)} \leq C\sum_{|I|\leq k}||\partial_y^I f||_{L^q(\Omega)}. \tag{2.A.44}$$

Proof. The estimates (2.A.42) and (2.A.44) are the usual Sobolev inequalities. The estimate (2.A.43) follows after applying the one-dimensional Sobolev inequality in the radial direction and the two-dimensional Sobolev inequality in the tangential directions. □

We also have the following product rule:

Lemma 2.A.6. *Suppose that $|\partial_y^I D_t^k f| \leq K$ in Ω for all $|I| + k \leq 3$. Then, if $k + \ell = s$, we have:*

$$||fg||_{k,\ell} \leq (||f||_{k,\ell} + K)(||g||_{k,\ell} + ||g||_{s-1}). \tag{2.A.45}$$

The right-hand side can also be bounded by $(||f||_s + L)||g||_s$, but for some our applications it is more useful to keep track of which types of derivatives land on f.

Proof. We need to bound $||(D_t^{k_1}\partial_y^{J_1} f)(D_t^{k_2}\partial_y^{J_2} g)||_{L^2(\Omega)}$ where $k_1+k_2+|J_1|+|J_2|=s$. If $k_1+|J_1|\leq 3$, we bound this by $||D_t^{k_1}\partial_y^{J_1} f||_{L^\infty(\Omega)}||D_t^{k_2}\partial_y^{J_2} g||_{L^2(\Omega)}$ which is bounded by the right-hand side of (2.A.45). If instead $k_1+|J_1| \geq 4$, we bound it by $||D_t^{k_1}\partial_y^{J_1} f||_{L^2(\Omega)}||D_t^{k_2}\partial_y^{J_2} g||_{L^\infty(\Omega)} \leq ||f||_{k,\ell}||g||_{2+k_2+|J_2|}$. Since $k_1 + |J_1| \geq 4$ and $k_1 + k_2 + |J_1| + |J_2| = s$, it follows that $2 + k_2 + |J_2| \leq s$, as required. □

2.A.4 The extension operator

Fix an integer $s \geq 0$. Let $\eta = \eta(r)$ be a smooth cutoff function which is one when $r \leq 1 + 1/(4+4s)$ and zero when $r \geq 1 + 1/(2+2s)$. Let $\lambda_0, ..., \lambda_s$ be the solution to the system $\sum_{j=0}^{s} \lambda_j(-(j+1))^\ell = 1$ for $\ell = 0, ..., s$. If $f : \Omega \to \mathbb{R}$, we extend f to a function $Ef = E_s f$ on $\mathbb{R}^3$ by setting $Ef(y) = f(y)$ when $|y| \leq 1$ and when $|y| \geq 1$, write $f(y) = f(r, \omega)$ where $r = |y|, \omega = y/|y| \in \mathbb{S}^2$ and define:

$$Ef(r,\omega) = \sum_{j=0}^{s} \lambda_j f(r - (j+1)(r-1), \omega)\eta(r), \quad r \geq 1. \tag{2.A.46}$$

Let $\zeta = \zeta(r)$ be a smooth function with $\zeta(r) = 0, r \leq 1/4$ and $\zeta = 1$ for $r \geq 1/2$. For $f : \mathbb{R}^3 \to \mathbb{R}$, we define $||f||^2_{H^{(k,s)}(\mathbb{R}^3)} = \sum_{|I| \leq k} \int_0^\infty ||\partial_y^I(\zeta f)(r, \cdot)||^2_{H^s(\partial\Omega)} r^2 dr + ||(1-\zeta)f||^2_{H^{k+s}(\Omega)}$, and we have:

Theorem 2.A.1. *Fix $s \geq 2$ and define $E = E_s$ by (2.A.46). Then E is continuous as a map $H^s(\Omega) \to H^s(\mathbb{R}^3)$ and $H^{(s,1/2)}(\Omega) \to H^{(s,1/2)}(\mathbb{R}^3)$ and there are constants $0 < C_1 < C_2 < \infty$ depending only on s so that*

$$C_1 ||Ef||_{H^{(s,a)}(\mathbb{R}^3)} \leq ||f||_{H^{(s,a)}(\Omega)} \leq C_2 ||Ef||_{H^{(s,a)}(\mathbb{R}^3)} \qquad \text{where } a = 0, 1/2, \tag{2.A.47}$$

there is a constant C depending only on s so that if T is any vector field on $\mathbb{R}^3$ with $T|_\Omega \in \mathcal{T}$, then:

$$||TEf||_{H^{(s,a)}(\mathbb{R}^3)} \leq C(||ETf||_{H^{(s,a)}(\mathbb{R}^3)} + ||Ef||_{H^{(s,a)}(\mathbb{R}^3)}), \quad \text{where } a = 0, 1/2. \tag{2.A.48}$$

Proof. We have:

$$\partial_r^\ell(Ef)(r,\omega) = \sum_{j=0}^{s} \lambda_j \partial_r^\ell f(r - (j+1)(r-1), \omega)\big(-(j+1)\eta(r)\big)^\ell + g_\ell(r,\omega), \quad r \geq 1, \tag{2.A.49}$$

where $g_\ell(1,\omega) = 0$, so by the definition of the λ_j and the fact that $\eta(1) = 1$, it follows that $\partial_r^k(Ef)(1,\omega) = \partial_r^k f(1,\omega)$ for $1 \leq k \leq s$ and $\omega \in \mathbb{S}^2$. This implies the estimate (2.A.47). The estimate (2.A.48) follows from the fact that near the boundary, $T \in \mathcal{T}$ commutes with E since $(y^a \partial_b - \partial_b y^a)|y|^2 = 0$. $\square$

2.A.5 The Green's formula

We conclude this section by recording the following Green's formula which will be frequently used throughout this manuscript. Let $f, g : \mathcal{D} \to \mathbb{R}$ be C^1 functions, then:

$$\int_\Omega \widetilde{\partial}_i f(\widetilde{x}(t,y)) g(\widetilde{x}(t,y)) \widetilde{\kappa} dy = \int_{\widetilde{\mathcal{D}}_t} \widetilde{\partial}_i f(\widetilde{x}) g(\widetilde{x}) d\widetilde{x}$$

$$= -\int_{\widetilde{\mathcal{D}}_t} f(\widetilde{x}) \widetilde{\partial}_i g(\widetilde{x}) d\widetilde{x} + \int_{\partial\widetilde{\mathcal{D}}_t} N_i f(\widetilde{x}) g(\widetilde{x}) dS(\widetilde{x})$$

$$= -\int_\Omega f(y) \widetilde{\partial}_i g(y) \widetilde{\kappa} dy + \int_{\partial\Omega} N_i f(y) g(y) \widetilde{\nu} dS(y). \quad (2.A.50)$$

2.B Proofs of Elliptic estimates for the Dirichlet Problem

Here we prove the elliptic estimates we need. We will use these to prove that Λ is a continuous map on a certain Banach space and to prove that Λ is a contraction, in Section 2.9. The basic estimates we need for the contraction estimates imply the estimates for the operator norm so we start with the contraction estimates.

Let $V_I, V_{II} : [0,T] \times \Omega \to \mathbb{R}^3$ be two vector fields on Ω and let $\widetilde{x}_I, \widetilde{x}_{II}$ denote their smoothed flows (2.4.1). Set

$$A_{I\,a}^{\,i} = \frac{\partial \widetilde{x}_I^i}{\partial y^a}, \qquad A_{I\,i}^{\,a} = \frac{\partial y^a}{\partial \widetilde{x}_I^i} \qquad \text{and} \qquad A_{II\,a}^{\,i} = \frac{\partial \widetilde{x}_{II}^i}{\partial y^a}, \qquad A_{II\,i}^{\,a} = \frac{\partial y^a}{\partial \widetilde{x}_{II}^i}. \quad (2.B.1)$$

We will assume that:

$$\textstyle\sum_{k+|J|\le 3} |\partial_y^J \widetilde{x}_I| + |\partial_y^J \widetilde{x}_{II}| \le M_0. \quad (2.B.2)$$

By the formula for the derivative of the inverse (2.D.2) this implies that $|A_{I\,i}^{\,a}| + |A_{II\,i}^{\,a}| \le C(M_0)$.

We define

$$\widetilde{\partial}_{Ii} = A_{I\,i}^{\,a}\frac{\partial}{\partial y^a}, \qquad \text{and} \qquad \widetilde{\partial}_{IIi} = A_{II\,i}^{\,a}\frac{\partial}{\partial y^a}, \tag{2.B.3}$$

$$\widetilde{g}_I^{ab} = \delta^{ij}A_{I\,i}^{\,a}A_{I\,j}^{\,b}, \qquad \text{and} \qquad \widetilde{g}_{II}^{ab} = \delta^{ij}A_{II\,i}^{\,a}A_{II\,j}^{\,b}, \tag{2.B.4}$$

as well as:

$$\widetilde{\Delta}_I f = \delta^{ij}\widetilde{\partial}_{Ii}\widetilde{\partial}_{Ij} f = \partial_a\big(\widetilde{g}_I^{ab}\partial_b f\big), \qquad \widetilde{\Delta}_{II} f = \delta^{ij}\widetilde{\partial}_{Ii}\widetilde{\partial}_{IIj} f = \partial_a\big(\widetilde{g}_{II}^{ab}\partial_b f\big). \tag{2.B.5}$$

We define:

$$\mathrm{div}_{\mathrm{I}}\,\alpha = \delta^{ij}\widetilde{\partial}_{Ii}\alpha_j, \qquad \mathrm{div}_{\mathrm{II}}\,\alpha = \delta^{ij}\widetilde{\partial}_{IIi}\alpha_j, \tag{2.B.6}$$

$$(\mathrm{curl}_{\mathrm{I}}\,\alpha)_{ij} = \widetilde{\partial}_{Ii}\alpha_j - \widetilde{\partial}_{Ij}\alpha_i, \qquad (\mathrm{curl}_{\mathrm{II}}\,\alpha)_{ij} = \widetilde{\partial}_{IIi}\alpha_j - \widetilde{\partial}_{IIj}\alpha_i, \tag{2.B.7}$$

and

$$\gamma_I^{ij} = A_{I\,a}^{\,i}A_{I\,b}^{\,j}\gamma^{ab}, \qquad \text{and} \qquad \gamma_{II}^{ij} = A_{II\,a}^{\,i}A_{II\,b}^{\,j}\gamma^{ab}. \tag{2.B.8}$$

Here, we are writing γ^{ab} for the cometric on $\partial\Omega$ extended to the interior of Ω. Fixing a smooth radial function χ with $\chi(r) = 0$ for $r \leq \frac{1}{2}$ and $\chi(r) = 1$ for $r \geq \frac{3}{4}$, then:

$$\gamma^{ab} = \delta^{ab} - \chi(r)N^a N^a, \tag{2.B.9}$$

with N the unit normal to $\partial\Omega$.

Recalling the notation $\mathcal{T}^r$ from Section 2.3.3, we will use the following norms:

$$||\alpha||^2_{H^k(\Omega)} = \sum_{|J|\leq k}\int_\Omega \delta^{ij}\partial_y^J\alpha_i\partial_y^J\alpha_j\,dy, \qquad ||\alpha||_{C^k(\Omega)} = \sum_{\ell=0}^k ||\partial_y^\ell\alpha||_{L^\infty(\Omega)}, \tag{2.B.10}$$

$$||\mathcal{T}^r\alpha||^2_{L^2(\Omega)} = \int_\Omega |\mathcal{T}^r\alpha|^2 dy. \tag{2.B.11}$$

In what follows we will use the convention that the components of α will be expressed in terms of the $\widetilde{x}_I$ frame and β will be expressed in terms of the $\widetilde{x}_{II}$ frame and we will just write α, β instead of α_I, β_{II}. We now list the elliptic estimates we use. Proofs can be found in the following sections.

Lemma 2.B.1. *With the above definitions, if α, β are (0,1)-tensors on Ω then on $[0, T] \times \Omega$:*

$$|\widetilde{\partial}_I \alpha - \widetilde{\partial}_{II} \beta| \leq C(M')\big(|\mathrm{div_I}\, \alpha - \mathrm{div_{II}}\, \beta| + |\mathrm{curl_I}\, \alpha - \mathrm{curl_{II}}\, \beta| + |\mathcal{T}\alpha - \mathcal{T}\beta| \\ + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^1(\Omega)} |\widetilde{\partial}_{II}\beta|\big). \quad (2.B.12)$$

There is a higher-order version of Lemma 2.B.1 in Sobolev spaces and with mixed space and time derivatives:

Lemma 2.B.2. *Fix $r \geq 7$ and let $1 \leq \ell \leq r$. Suppose $\widetilde{x}_I, \widetilde{x}_{II} \in H^r(\Omega)$ satisfy (2.B.2). If $\alpha - \beta \in H^\ell_{loc}(\Omega)$ and:*

$$\mathrm{div_I}\, \alpha - \mathrm{div_{II}}\, \beta,\ \mathrm{curl_I}\, \alpha - \mathrm{curl_{II}}\, \beta\ \in H^{\ell-1}(\Omega), \quad (2.B.13)$$

$$T(\alpha - \beta) \in H^{\ell-1}(\Omega) \text{ for all } T \in \mathcal{T}, \quad (2.B.14)$$

$$\widetilde{\partial}_{II}\beta \in H^{\ell-1}(\Omega), \quad (2.B.15)$$

then $\alpha - \beta \in H^\ell(\Omega)$ and there is a constant $C_r = C_r(M_0, ||\widetilde{x}_I||_{H^r(\Omega)}, ||\widetilde{x}_{II}||_{H^r(\Omega)})$, so that

$$||\alpha - \beta||_{H^\ell(\Omega)} \leq C_r\big(||\,\mathrm{div_I}\, \alpha - \mathrm{div_{II}}\, \beta||_{H^{\ell-1}(\Omega)} + ||\,\mathrm{curl_I}\, \alpha - \mathrm{curl_{II}}\, \beta||_{H^{\ell-1}(\Omega)} \\ + ||\mathcal{T}^{\ell-1}(\widetilde{\partial}_I \alpha - \widetilde{\partial}_{II}\beta)||_{L^2(\Omega)} + (||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^\ell(\Omega)})||\widetilde{\partial}_{II}\beta||_{H^\ell(\Omega)}\big). \quad (2.B.16)$$

Similarly, if $k+\ell = s \le r$, $D_t^{k'}\widetilde{\partial}\beta \in H^{\ell'}(\Omega)$ for any $k'+\ell' \le s$ and:

$$D_t^k(\mathrm{div}_{\mathrm{I}}\alpha - \mathrm{div}_{\mathrm{II}}\,\beta), D_t^k(\mathrm{curl}_{\mathrm{I}}\alpha - \mathrm{curl}_{\mathrm{II}}\,\beta) \in H^{\ell-1}(\Omega), \tag{2.B.17}$$

$$D_t^k T(\alpha-\beta) \in H^{\ell-1}(\Omega),\ \textit{for all } T \in \mathcal{T}, \tag{2.B.18}$$

then $D_t^k(\alpha-\beta) \in H^\ell(\Omega)$ and there is a constant $C_r' = C_r'(M_0, ||\widetilde{x}_I||_r, ||\widetilde{x}_{II}||_r)$, so that:

$$\begin{aligned}&||\alpha-\beta||_{k,\ell} \le C_s'\big(||(\mathrm{div}_{\mathrm{I}}\,\alpha - \mathrm{div}_{\mathrm{II}}\,\beta)||_{k,\ell-1} + ||\,\mathrm{curl}_{\mathrm{I}}\,\alpha - \mathrm{curl}_{\mathrm{II}}\,\beta||_{k,\ell-1}\\ &+||\mathfrak{D}^{k,\ell-1}(\widetilde{\partial}_I\alpha - \widetilde{\partial}_{II}\beta)||_{L^2} + ||\alpha-\beta||_{s,0} + (||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_s)(||\beta||_{s,0} + ||\widetilde{\partial}\beta||_{s-1})\big).\end{aligned} \tag{2.B.19}$$

In the special case that $\alpha = \partial f, \beta = \partial g$ for functions $f, g \in H_0^1(\Omega)$, $\widetilde{\partial}_I f - \widetilde{\partial}_{II} g \in H^\ell_{loc}(\Omega)$, we have:

Proposition 2.B.1. *Suppose $\widetilde{x}_I, \widetilde{x}_{II} \in H^s(\Omega)$, $s \ge 1$, satisfy (2.B.2), $f-g \in H_0^1(\Omega)$, $\widetilde{\partial}_I f - \widetilde{\partial}_{II} g \in H^s_{loc}(\Omega)$ and that:*

$$\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g \in H^{s-1}(\Omega), \quad \widetilde{\partial}_{II} g \in H^s(\Omega), \quad T^J(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) \in L^2(\Omega),\ \textit{for all } |J| \le s. \tag{2.B.20}$$

Then $\widetilde{\partial}_I f - \widetilde{\partial}_{II} g \in H^s(\Omega)$ and there is a constant $C_s = C_s(M_0, ||\widetilde{x}_I||_{H^s(\Omega)}, ||\widetilde{x}_{II}||_s)$ so that

$$\begin{aligned}||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{H^s(\Omega)} &\le C_s||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{H^{s-1}(\Omega)} + C_s||\mathcal{T}(\widetilde{x}_I - \widetilde{x}_{II})||_{H^s(\Omega)}||\widetilde{\partial}_{II} g||_{H^s(\Omega)}\\ &+ C_s||\mathcal{T}\widetilde{x}_I||_{H^s(\Omega)}\big(||\widetilde{x}_I - \widetilde{x}_{II}||_{H^s(\Omega)}||\widetilde{\partial}_{II} g||_{H^{s-1}(\Omega)} + ||f-g||_{L^2}\big).\end{aligned} \tag{2.B.21}$$

Similarly, if $k+\ell = s$, the assumption (2.D.21) holds, $D_t^k(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) \in H^\ell_{loc}(\Omega)$ and:

$$D_t^k(\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g) \in H^{\ell-1}(\Omega), \tag{2.B.22}$$

$$T^J(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) \in L^2(\Omega), \text{ for all } T^J \in \mathfrak{D}^s, \tag{2.B.23}$$

$$D_t^k \widetilde{\partial}_{II} g \in H^\ell(\Omega), \tag{2.B.24}$$

then $D_t^k(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) \in H^\ell(\Omega)$ and there are constants $C_s' = C_s'(M, ||\widetilde{x}_I||_s, ||\widetilde{x}_{II}||_s)$ so that if $k+\ell = s$:

$$\begin{aligned} &||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{k,\ell} \\ &\quad \leq C_s'\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{k-1,\ell} + ||f-g||_{s+1,0} + ||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{s-1,1} + ||\mathcal{T}\widetilde{x}_{II}||_s ||f-g||_s\big) \\ &\qquad + C_r'||\mathcal{T}(\widetilde{x}_I - \widetilde{x}_{II})||_s\big(||\widetilde{\partial}_{II} g||_s + ||g||_{s+1,0}\big). \end{aligned} \tag{2.B.25}$$

We also need a result to build regularity for a function f with $\widetilde{\Delta} f \in H^{\ell-1}(\Omega)$ but with a priori only $f \in H^1_0(\Omega)$. Note that we are *not* assuming that $f \in H^\ell_{loc}(\Omega)$. This result is needed to prove a local-wellposedness result for the wave equation (2.4.7)-(2.4.8) (see Appendix 2.F.1). Writing $\widetilde{x} = \widetilde{x}_I$, we have:

Proposition 2.B.2. *Suppose $\widetilde{x} \in H^r(\Omega)$, $r \geq 5$, satisfies (2.B.2). If $f \in H^1_0(\Omega)$ and $\widetilde{\Delta} f \in H^{\ell-1}(\Omega)$ for some $0 \leq \ell \leq r$, then $\widetilde{\partial} f \in H^\ell(\Omega)$ and*

$$||\widetilde{\partial} f||_{H^\ell(\Omega)} \leq C(M_0, ||\widetilde{x}||_{H^r(\Omega)})\big(||\widetilde{\Delta} f||_{H^{\ell-1}(\Omega)} + ||\mathcal{T}\widetilde{x}||_{H^r(\Omega)}||f||_{L^2(\Omega)}\big). \tag{2.B.26}$$

Similarly, if $f \in H^1_0(\Omega)$, $D_t^k f \in L^2(\Omega)$ and $D_t^k \widetilde{\Delta} f \in H^{\ell-1}(\Omega)$, then $D_t^k \widetilde{\partial} f \in H^\ell(\Omega)$ and

$$||D_t^k \widetilde{\partial} f||_{H^\ell(\Omega)} \leq C(M_0, ||\widetilde{x}||_r)\big(||D_t^k \widetilde{\Delta} f||_{H^{\ell-1}(\Omega)} + ||\mathcal{T}\widetilde{x}||_r ||D_t^k f||_{L^2(\Omega)}\big). \tag{2.B.27}$$

We also need estimates which involve fractional derivatives on $\partial\Omega$.

Proposition 2.B.3. *Let α be a vector field on Ω. Fix $r \geq 5$. Then, for $1 \leq \ell \leq r$, there are continuous functions $C_\ell = C_\ell\big(M_0, ||\widetilde{x}||_{H^r(\Omega)}\big)$ so that*

$$||\alpha||^2_{H^\ell} \leq C_\ell\Big(||\mathrm{div}\,\alpha||^2_{H^{\ell-1}} + ||\mathrm{curl}\,\alpha||^2_{H^{\ell-1}} + ||\alpha||^2_{H^1} + \sum_{\mu=1}^N \int_{\partial\Omega} (\langle\partial_\theta\rangle_\mu^{1/2} \mathcal{T}^{r-1}\alpha^i)\cdot(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\alpha^j) N_i N_j dS\Big), \quad (2.B.28)$$

$$||\alpha||^2_{H^\ell} \leq C_\ell\Big(||\mathrm{div}\,\alpha||^2_{H^{\ell-1}} + ||\mathrm{curl}\,\alpha||^2_{H^{\ell-1}} + ||\alpha||^2_{H^1} + \sum_{\mu=1}^N \int_{\partial\Omega} (\langle\partial_\theta\rangle_\mu^{1/2} \mathcal{T}^{r-1}\alpha^i)\cdot(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\alpha^j) \gamma_{ij} dS\Big). \quad (2.B.29)$$

We will need the following lemma to exchange normal and tangential components of vector fields on $\partial\Omega$. This estimate appears in Lemma 5.6 of [11].

Lemma 2.B.3. *If α is a (0,1)-tensor on Ω and γ denotes the metric on $\partial\Omega_t$, then:*

$$\Big|\int_{\partial\Omega} (\gamma^{ij} - N^i N^j)\alpha_i\alpha_j d\mu_\gamma\Big| \leq \big(||\,\mathrm{div}\,\alpha||_{L^2(\Omega)} + ||\,\mathrm{curl}\,\alpha||_{L^2(\Omega)} + K||\alpha||_{L^2(\Omega)}\big)||\alpha||_{L^2(\Omega)}. \quad (2.B.30)$$

Finally, in Section 2.F.1, we will need the following elliptic estimate in $H^2(\Omega)$:

Lemma 2.B.4. *Let $\Delta_y = \partial^2_{y^1} + \partial^2_{y^2} + \partial^2_{y^3}$ be the flat Laplacian in the y coordinates. If $f \in H^1_0(\Omega) \cap H^2(\Omega)$, then:*

$$||\widetilde{\partial} f||_{H^1(\Omega)} \leq C(M)\big((\Delta_y f, \widetilde{\Delta} f)_{L^2(\Omega)} + ||\widetilde{\partial} f||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\big). \quad (2.B.31)$$

Proof of Lemma 2.B.1

The case with $\beta = 0$ is Lemma 5.5 in [11], and this version is Lemma B.4.1 of [2]. For the reader's convenience, we include the proof here. We start by setting:

$$\big(\operatorname{def}_I \alpha\big)_{ij} = \widetilde{\partial}_{Ii}\alpha_{Ij} + \widetilde{\partial}_{Ij}\alpha_{Ii}, \quad (D_I\alpha)_{ij} = \operatorname{div}_I \alpha\delta_{ij}, \quad (\widehat{D_I}\alpha)_{ij} = \big(\operatorname{def}_I \alpha - \frac{2}{3}D_I\alpha\big)_{ij}, \tag{2.B.32}$$

with a similar definition for $\operatorname{def}_{II}, D_{II}$, and $\widehat{D_{II}}$. We write:

$$\widetilde{\partial}_I\alpha - \widetilde{\partial}_{II}\beta = \frac{1}{3}\big(D_I\alpha - D_{II}\beta\big) + \frac{1}{2}\big(\operatorname{curl}_I \alpha - \operatorname{curl}_{II} \beta\big) + \frac{1}{2}\big(\widehat{D_I}\alpha - \widehat{D_{II}}\beta\big). \tag{2.B.33}$$

The first and second terms are bounded by the right-hand side of (2.B.12), and we now show how to control the last term. Let $S_{ij} = (\widehat{D_I}\alpha - \widehat{D_{II}}\beta)_{ij}$. Writing $\delta^{ij} = \gamma_I^{ij} - N_I^i N_I^j$ and using that S is symmetric, we have:

$$\delta^{ij}\delta^{k\ell}S_{ik}S_{j\ell} = \big(\gamma_I^{ij}\gamma_I^{k\ell} + 2\gamma_I^{ij}N_I^{k\ell} + N_I^i N_I^j N_I^k N_I^\ell\big)S_{ik}S_{j\ell}. \tag{2.B.34}$$

Now, because $\delta^{ij}S_{ij} = 0$, the last term is:

$$\big(N_I^i N_I^j S_{ij}\big)^2 = \big(\delta^{ij}S_{ij} - \gamma_I^{ij}S_{ij}\big)^2 = \big(\gamma_I^{ij}S_{ij}\big)^2 \leq 2\gamma_I^{ij}\gamma_I^{k\ell}S_{ik}S_{j\ell}, \tag{2.B.35}$$

where we have used that if T is a symmetric matrix then $(\operatorname{tr} T)^2 \leq \operatorname{rank} T \operatorname{tr}(T^2)$. Returning to (2.B.34), we have:

$$|S|^2 \leq 2\gamma_I^{ij}\big(\gamma_I^{k\ell} + N_I^k N_I^\ell\big)S_{ik}S_{j\ell} = 2\gamma_I^{ij}\delta^{k\ell}S_{ik}S_{j\ell}. \tag{2.B.36}$$

We now write:

$$S_{ij} = \big(\operatorname{def}_I \alpha - \operatorname{def}_{II} \beta\big)_{ij} - \frac{2}{3}\big(D_I\alpha - D_{II}\beta\big)_{ij} \equiv S_{ij}^1 + S_{ij}^2. \tag{2.B.37}$$

Since $|S^2| \leq C|\operatorname{div}_I \alpha - \operatorname{div}_{II} \beta|$, it suffices to control S^1. We have:

$$\begin{aligned}
&\gamma_I^{ij}\delta^{k\ell}S^1_{ik}S^1_{j\ell} \\
&= \gamma_I^{ij}\delta^{k\ell}\big(\widetilde{\partial}_{Ii}\alpha_{Ik} - \widetilde{\partial}_{IIi}\beta_{IIk} + \widetilde{\partial}_{Ik}\alpha_{Ii} - \widetilde{\partial}_{IIk}\beta_{IIi}\big)\big(\widetilde{\partial}_{Ij}\alpha_{I\ell} - \widetilde{\partial}_{IIj}\beta_{II\ell} + \widetilde{\partial}_{I\ell}\alpha_{Ij} - \widetilde{\partial}_{II\ell}\beta_{IIj}\big).
\end{aligned} \quad (2.B.38)$$

To bound the product of the first term in the first factor with the first term in the second factor, we replace $\widetilde{\partial}_{II}\beta_{II}$ with $\widetilde{\partial}_{I}\beta_{II}$, which generates terms that are bounded by the last term on the right-hand side of (2.B.12). The resulting term only involves tangential derivatives of α, β but these are with respect to $\widetilde{x}_I$. However we can replace these with tangential derivatives with respect to y up to terms that are bounded by the last term on the right-hand side of (2.B.12). For the product of the second term in the first factor and the second term in the second factor we instead note that it can be controlled in terms of $|\operatorname{curl}_I\alpha - \operatorname{curl}_{II}\beta|^2$ along with the third and fourth terms on the right-hand side of (2.B.12) The other terms in (2.B.38) can be handled similarly.

Proof of Lemma 2.B.2

Both estimates have essentially the same proof, so we will just prove the second. The first one follows from the same argument, but one uses the commutator estimate 2.D.4 with $\mathcal{U} = \{\partial_{y^1}, \partial_{y^2}, \partial_{y^3}\}$ instead of $\mathcal{U} = \mathcal{D}$. The only difference is that in the proof of (2.B.16) no time derivatives enter.

We argue by induction. When $s = 1$, the result follows from the pointwise estimate after writing:

$$\partial_a(\alpha - \beta) = A_{I\,a}^{\,i}(\widetilde{\partial}_{Ii}\alpha - \widetilde{\partial}_{IIi}\beta) + (A_{I\,a}^{\,i} - A_{II\,a}^{\,i})\widetilde{\partial}_{Ii}\beta. \quad (2.B.39)$$

We now assume that we have the result for $s \leq m-1$. We write $T^I = D_t^k\partial_y^J \in \mathcal{D}^{k,\ell}$ where

$k+|J| = m$. If $|J| = 0$ there is nothing to prove, so we consider $|J| \geq 1$. We then write $D_t^k \partial_y^J = \partial_a D_t^k \partial_y^{J'}$ where $J = (a, J')$ and $\partial_a = A_{I\,a}^{\,i} \widetilde{\partial}_i$. Applying the pointwise estimate (2.B.12) and integrating over an arbitrary $U \subset\subset \Omega$, we have:

$$\begin{aligned} &||T^I(\alpha-\beta)||_{L^2(U)} \\ &\quad \leq C(M_0)\big(||\,\mathrm{div}_{\mathrm{I}}\, D_t^k \partial_y^{J'} \alpha - \mathrm{div}_{\mathrm{II}}\, D_t^k \partial_y^{J'} \beta||_{L^2(\Omega)} + ||\,\mathrm{curl}_{\mathrm{I}}\, D_t^k \partial_y^{J'} \alpha - \mathrm{curl}_{\mathrm{II}}\, D_t^k \partial_y^{J'} \beta||_{L^2(\Omega)} \\ &\qquad + ||\mathcal{T} D_t^k \partial_y^{J'}(\alpha-\beta)||_{L^2(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)} ||\widetilde{\partial}_{II} D_t^k \partial_y^{J'} \beta||_{L^2(\Omega)}\big). \end{aligned} \tag{2.B.40}$$

Using the commutator estimate from Lemma 2.D.4 with $\mathcal{U} = \mathcal{D}$, the last term is bounded by the right-hand side of (2.B.19). To deal with the first two terms, we apply the commutator estimate (2.D.23) with $\mathcal{U} = \mathcal{D}$:

$$\begin{aligned} &||\,\mathrm{div}_{\mathrm{I}} D_t^k \partial_y^{J'} \alpha - \mathrm{div}_{\mathrm{II}} D_t^k \partial_y^{J'} \beta||_{L^2} \\ &\quad \leq ||D_t^k \partial_y^{J'}(\mathrm{div}_{\mathrm{I}} \alpha - \mathrm{div}_{\mathrm{II}}\, \beta)||_{L^2} + C_s\big(||\widetilde{\partial}_I \alpha - \widetilde{\partial}_{II} \beta||_{m-2} + ||\widetilde{x}_I - \widetilde{x}_{II}||_s ||\widetilde{\partial}_{II} \beta||_{m-2}\big), \end{aligned} \tag{2.B.41}$$

where $L^2 = L^2(\Omega)$ and $C_s = C_s(M, ||\widetilde{x}_I||_s, ||\widetilde{x}_{II}||_s)$, along with a similar estimate for the curl. All of these terms are bounded by the right-hand side of (2.B.19). To deal with the last term on the right-hand side of (2.B.40), we commute the tangential derivative with $D_t^k \partial_y^{J'}$:

$$|\mathcal{T} D_t^k \partial_y^{J'}(\alpha-\beta)| \leq \textstyle\sum_{T \in \mathcal{T}} |D_t^k \partial_y^{J'} T(\alpha-\beta)| + C|D_t^k \partial_y^{J'}(\alpha-\beta)|. \tag{2.B.42}$$

The second term here is bounded by the right-hand side of (2.B.19) by the inductive assumption. To control the first term in L^2, we apply the inductive assumption with α, β replaced by

$T\alpha, T\beta$, and this gives:

$$||D_t^k \partial_y^{J'} T(\alpha-\beta)||_{L^2(\Omega)}$$

$$\leq C_s || \operatorname{div}_{\mathrm{I}} T\alpha - \operatorname{div}_{\mathrm{II}} T\beta||_{k,\ell-2} + || \operatorname{curl}_{\mathrm{I}} T\alpha - \operatorname{curl}_{\mathrm{II}} T\beta||_{k,\ell-2} + ||\mathfrak{D}^{k,\ell}(\alpha-\beta)||_{L^2(\Omega)}$$

$$+ C_s(||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)} + ||\widetilde{x}_I - \widetilde{x}_{II}||_r)||\widetilde{\partial}_{II} T\beta||_{m-1}. \quad \text{(2.B.43)}$$

We now write $\operatorname{div} T(\alpha-\beta) = T \operatorname{div}(\alpha-\beta) - TA_{I\,i}^{\,a}\partial_a(\alpha^i-\beta^i)$, and use the product rule (2.A.45) and (2.D.2):

$$||(TA_{I\,i}^{\,a})\partial_a(\alpha^i-\beta^i)||_{k,\ell-2} \leq C(M, ||\widetilde{x}_I||_s)||\alpha-\beta||_{m-1}. \quad \text{(2.B.44)}$$

Arguing as with the other terms in (2.B.43), recalling that we are integrating over any $U \subset\subset \Omega$ gives the result.

Proof of Proposition 2.B.1

To motivate the proof, first consider the case that $\widetilde{x}_{II} = \widetilde{x}_I$ and $g = 0$. If $\widetilde{x}_I$ was smooth, one could get a version of this estimate without tangential derivatives by straightening the boundary and using a standard integration by parts argument. Because the coordinate $\widetilde{x}_I$ is only smooth in tangential directions, the idea is instead to first use the estimate (2.B.16) to replace the derivatives of $\widetilde{\partial} f$ with derivatives of Δf and tangential derivatives of $\widetilde{\partial} f$, and then apply the integration by parts argument to this. One then has to deal with commutators $[\mathcal{T}^r, \widetilde{\partial}]f$. To highest order, this behaves like $(\mathcal{T}^r \partial_y \widetilde{x}_I)\partial_y f$, and because the derivatives $\mathcal{T}$ are tangential this term can be handled. Also note that since $\mathcal{T}^r f = 0$ on $\partial\Omega$, the boundary terms that arise when integrating by parts vanish so we avoid the need to straighten the boundary.

We start with the following estimate:

Lemma 2.B.5. *Under the hypotheses of Proposition 2.B.1, we have:*

$$||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{L^2}^2 \leq C(M_0)\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{L^2}^2 + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)}^2 ||\widetilde{\partial}_{II} g||_{L^2}^2\big). \tag{2.B.45}$$

Proof. We write $\widetilde{\partial}_{II} g = \widetilde{\partial}_I g + (A_{II} - A_I) \cdot \partial_y g$ and since $||\alpha||_{L^2(\Omega)}^2$ is comparable to $\int_\Omega |\alpha|^2 \widetilde{\kappa}\, dy$:

$$||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{L^2(\Omega)}^2 \lesssim \int_\Omega \delta^{ij} \big(\widetilde{\partial}_{Ii} f - \widetilde{\partial}_{IIi} g\big)\big(\widetilde{\partial}_{Ij} f - \widetilde{\partial}_{IIj} g\big) \widetilde{\kappa}_I dy$$

$$= \int_\Omega \delta^{ij} \big(\widetilde{\partial}_{Ii} f - \widetilde{\partial}_{IIi} g\big)\big(\widetilde{\partial}_{Ij} f - \widetilde{\partial}_{IIj} g\big) \widetilde{\kappa}_I dy + 2 \int_\Omega \delta^{ij} (A_{I\,i}^{\,a} - A_{II\,i}^{\,a}) A_{I\,j}^{\,b} (\partial_a g) \partial_b (f - g) \widetilde{\kappa}_I\, dy$$

$$+ \int_\Omega \delta^{ij} (A_{I\,i}^{\,a} - A_{II\,i}^{\,a})(A_{I\,j}^{\,b} - A_{II\,j}^{\,b})(\partial_a g)(\partial_b g) \widetilde{\kappa}_I dy. \tag{2.B.46}$$

The terms on the last line are bounded by the second term on the right-hand side of (2.B.45), using Lemma 2.D.2 and Sobolev embedding. To control the terms on the first line, we integrate by parts:

$$\int_\Omega \delta^{ij} A_{I\,i}^{\,a} A^a_{\,i} \partial_a (f-g) A_{I\,j}^{\,b} \partial_b (f-g) \widetilde{\kappa}_I dy = -\int_\Omega (f-g) \frac{1}{\widetilde{\kappa}_I} \partial_a \big(\widetilde{\kappa}_I \delta^{ij} A_{I\,i}^{\,a} A_{I\,j}^{\,b} \partial_b (f-g)\big) \widetilde{\kappa}_I dy. \tag{2.B.47}$$

The second factor here is $\widetilde{\Delta}_I (f-g) = \big(\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g\big) + (\widetilde{\Delta}_I - \widetilde{\Delta}_{II}) g$. Since we want a bound that only involves one derivative of g, we further write:

$$(\widetilde{\Delta}_I - \widetilde{\Delta}_{II}) g = \frac{1}{\widetilde{\kappa}_I} \partial_a \Big(\widetilde{\kappa}_I (\widetilde{g}_I^{ab} \partial_b g) - \widetilde{\kappa}_{II} (\widetilde{g}_{II}^{ab} \partial_b g)\Big) + \Big(\frac{1}{\widetilde{\kappa}_{II}} - \frac{1}{\widetilde{\kappa}_I}\Big) \partial_a \big(\widetilde{\kappa}_{II} \widetilde{g}_{II}^{ab} \partial_b g\big), \tag{2.B.48}$$

and then integrate by parts and use Poincarè's inequality again, which shows that:

$$\Big| \int_\Omega (f-g)(\widetilde{\Delta}_I - \widetilde{\Delta}_{II}) g \widetilde{\kappa} dy \Big| \leq C(M_0) ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)} ||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{L^2} ||\widetilde{\partial}_{II} g||_{L^2}. \qquad \square$$

We now consider the case $\alpha = \widetilde{\partial}_I f, \beta = \widetilde{\partial}_{II} g$ for functions $f, g \in H_0^1(\Omega)$. We then have:

Proposition 2.B.4. *With the hypotheses of Proposition 2.B.1, for each s there are constants*

$C_s = C_s(M, ||\widetilde{x}_I||_{H^s(\Omega)}, ||D_t\widetilde{x}_I||_s, ||\widetilde{x}_{II}||_{H^s(\Omega)}, ||D_t\widetilde{x}_{II}||_s)$ *so that if* $k+\ell = s$*:*

$$||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{k,\ell} \leq C_s\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{k-1,\ell} + ||f-g||_{s,0} + ||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{s-1,1}$$

$$+ ||\mathcal{T}\widetilde{x}_{II}||_s ||f-g||_s + ||\mathcal{T}(\widetilde{x}_I - \widetilde{x}_{II})||_s \big(||\widetilde{\partial}_{II} g||_s + ||g||_{s+1,0}\big)\big). \quad (2.B.49)$$

This proposition follows from (2.B.19) and the following lemma:

Lemma 2.B.6. *With the hypotheses as above, there is a constant* $C_s(M, ||\widetilde{x}_I||_s, ||\widetilde{x}_{II}||_s)$ *so that for any* $\delta > 0$*:*

$$||\mathfrak{D}^{k,\ell}(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)||_{L^2(\Omega)} \leq C_s\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||_{k-1,\ell} + \delta||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{k,\ell} +$$

$$\delta^{-1}||\mathcal{T}(\widetilde{x}_I - \widetilde{x}_{II})||_s ||\widetilde{\partial}_{II} g||_s + \delta^{-1}||\mathcal{T}\widetilde{x}_I||_s (||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{s,0} + ||f-g||_{s,0})\big), \quad (2.B.50)$$

for $s = k + \ell$*.*

Proof of Lemma 2.B.6. For the purposes of the below proof, the commutator $[T, \partial_a]$ for $T \in \mathcal{T}$ will be ignored for notational convenience. We argue by induction. When $s = 1$, we fix a multi-index I with $|I| = 1$. If $T^I = D_t$ there is nothing to prove so we assume that $T^I = S \in \mathcal{T}$. We start by writing:

$$||S\widetilde{\partial}_I f - S\widetilde{\partial}_{II} g||^2_{L^2(\Omega)}$$

$$= \int_\Omega (\widetilde{\partial}_I S f - \widetilde{\partial}_{II} S g) \cdot S(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)\, dy + \int_\Omega \big([\widetilde{\partial}_I, S] f - [\widetilde{\partial}_{II}, S] g\big) \cdot S(\widetilde{\partial}_I f - S\widetilde{\partial}_{II} g)\, dy. \quad (2.B.51)$$

To deal with the first term, we integrate by parts and use that $Sf = Sg = 0$ on $\partial\Omega$, which gives:

$$\int_\Omega (\widetilde{\partial}_I Sf - \widetilde{\partial}_{II} Sg) \cdot S(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) dy$$

$$= \int_\Omega Sf \partial_a \big(\delta^{ij} A_{I\,i}^{\,a} \{S\widetilde{\partial}_{Ij} f - S\widetilde{\partial}_{IIj} g\}\big) dy - \int_\Omega Sg\, \partial_a \big(\delta^{ij} A_{II\,i}^{\,a} \{S\widetilde{\partial}_{Ij} f - S\widetilde{\partial}_{IIj} g\}\big) dy. \quad (2.B.52)$$

We write the first term on the right-hand side as:

$$\int_\Omega \delta^{ij} \big((Sf) A_{I\,i}^{\,a} - (Sg) A_{II\,i}^{\,a}\big) \partial_a \big(S\widetilde{\partial}_{Ij} f - S\widetilde{\partial}_{IIj} g\big)\, dy$$

$$+ \int_\Omega \delta^{ij} \big(Sf \partial_a (A_{I\,i}^{\,a}) - Sg \partial_a (A_{II\,i}^{\,a})\big) \big(S\widetilde{\partial}_I f - S\widetilde{\partial}_{II} g\big)\, dy. \quad (2.B.53)$$

The second term here is bounded by the right-hand side of (2.B.50). We now re-write the first term as:

$$\int_\Omega (Sf) \widetilde{\partial}_I \cdot (S\widetilde{\partial}_I f - S\widetilde{\partial}_{II} g) - (Sg) \widetilde{\partial}_{II} \cdot (S\widetilde{\partial}_I f - S\widetilde{\partial}_{II} g)$$

$$= \int_\Omega (Sf) S(\widetilde{\Delta}_I f - \widetilde{\partial}_I \cdot \widetilde{\partial}_{II} g) - (Sg) S(\widetilde{\partial}_{II} \cdot \widetilde{\partial}_I f - \widetilde{\Delta}_{II} g)$$

$$+ \int_\Omega (Sf) [\widetilde{\partial}_I, S] \cdot (\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) - (Sg) [\widetilde{\partial}_{II}, S] \cdot (\widetilde{\partial}_I f - \widetilde{\partial}_{II} g). \quad (2.B.54)$$

Finally, we re-write the first term on the right-hand side as:

$$\int_\Omega (Sf - Sg) S(\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g) + \int_\Omega (Sf - Sg) S(\widetilde{\partial}_I - \widetilde{\partial}_{II}) \cdot \widetilde{\partial}_{II} g, \quad (2.B.55)$$

and integrate S by parts in each of these terms. Applying Cauchy's inequality, the result of the above is:

$$||S(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)||^2_{L^2(\Omega)}$$

$$\leq C_1\big(||\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g||^2_{L^2(\Omega)} + \delta^{-1}||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||^2_{L^2(\Omega)}\big) + C_1\delta^{-1}||\widetilde{x}_I - \widetilde{x}_{II}||_{C^2(\Omega)}||\widetilde{\partial}_{II} g||_{H^1(\Omega)}$$

$$+ C_1\delta\big(||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||^2_{H^1(\Omega)} + ||[\widetilde{\partial}_I, S]f - [\widetilde{\partial}_{II}, S]g||^2_{L^2(\Omega)} + ||[\widetilde{\partial}_I, S]\cdot\widetilde{\partial}_I f - [\widetilde{\partial}_{II}, S]\cdot\widetilde{\partial}_{II} g||^2_{L^2(\Omega)}\big). \tag{2.B.56}$$

Here, and in what follows, we will use C_k to denote a constant which depends on $M, ||\widetilde{x}_I||_k, ||\widetilde{x}_{II}||_k$. Applying the commutator estimate (2.D.23), every term here is bounded by the right-hand side of (2.B.50).

We now suppose we have the result for $s = 1, ..., m-1$, and fix $T^I = D_t^k T^J$ where $T^K \in \mathcal{T}^\ell$ with $k + \ell = m$. If $|K| = 0$ there is nothing to prove so we assume that $T^I = ST^J$ for some $S \in \mathcal{T}$ and $T^J \in \mathfrak{D}^{k,\ell-1}$. The proof now follows in nearly the same way as above, so we just indicate the main points. First, we write:

$$\int_\Omega T^I(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) T^I(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)\, dy = \int_\Omega (\widetilde{\partial}_I T^I f - \widetilde{\partial}_{II} T^I g) T^I(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) dy$$

$$+ \int_\Omega ([\widetilde{\partial}_I, T^I] f - [\widetilde{\partial}_{II}, T^I] g) T^I(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g) dy. \tag{2.B.57}$$

Integrating by parts in the first term yields, in addition to lower-order terms:

$$\int_\Omega (T^I f)\widetilde{\partial}_I \cdot (T^I \widetilde{\partial}_I f - T^I \widetilde{\partial}_{II} g) - (T^I g)\widetilde{\partial}_{II} \cdot (T^I \widetilde{\partial}_I f - T^I \widetilde{\partial}_{II} g)\, dy. \tag{2.B.58}$$

We now write $T^I = ST^J$ in the second factor in each term and then commute S with $\widetilde{\partial}_I, \widetilde{\partial}_{II}$, and obtain:

$$\int_\Omega (T^J f - T^J g) S(T^J \widetilde{\Delta}_I f - T^J \widetilde{\Delta}_{II} g) + \int_\Omega (T^J f - T^J g) ST^J\big((\widetilde{\partial}_I - \widetilde{\partial}_{II}) \cdot \widetilde{\partial}_{II} g\big). \tag{2.B.59}$$

Integrating S by parts and bounding:

$$||T^J\big((\widetilde{\partial}_I - \widetilde{\partial}_{II}) \cdot \widetilde{\partial}_{II} g\big)||_{L^2(\Omega)} \leq ||\widetilde{x}_I - \widetilde{x}_{II}||_r ||\widetilde{\partial}_{II} g||_m, \tag{2.B.60}$$

shows that $||T^I(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)||_{L^2(\Omega)}$ is bounded by:

$$C_m\big(||T^J(\widetilde{\Delta}_I f - \widetilde{\Delta}_{II} g)||_{L^2(\Omega)} + (1+\delta^{-1})||T^J(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)||^2_{L^2(\Omega)}$$

$$+ (1+\delta^{-1})||\mathcal{T}(\widetilde{x}_I - \widetilde{x}_{II})||^2_m ||\widetilde{\partial} g||^2_{k,\ell} + C_m \delta \Big\{ ||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||^2_{k,\ell}$$

$$+ ||[\widetilde{\partial}_I, S] T^J f - [\widetilde{\partial}_{II}, S] T^J g||_{L^2(\Omega)} + ||[\widetilde{\partial}_I, S] T^J \widetilde{\partial}_I f - [\widetilde{\partial}_I, S] T^J \widetilde{\partial}_{II} g||_{L^2(\Omega)} \Big\}. \tag{2.B.61}$$

The result now follows after using the commutator estimate (2.D.23) and induction. $\square$

Proof of Proposition 2.B.2

We just prove the $k = 0$ case, as the $k \geq 1$ case follows using similar arguments. This would be a consequence of the Proposition 2.B.1 with $g = 0$ if we knew that $\widetilde{\partial} f \in H^m_{loc}(\Omega)$ and $\mathcal{T}^I \widetilde{\partial} f \in L^2(\Omega)$ for all $|I| \leq m$. In the following lemma we prove that this is the case. See Section 2.A for the definitions of the sets U_α and the vector fields $T \in \mathcal{T}$.

Lemma 2.B.7. *Fix $s \geq 0$ and suppose that $\widetilde{x} \in H^s(\Omega), T\widetilde{x} \in H^s(\Omega)$ for all $T \in \mathcal{T}$ and that (2.B.2) holds. Suppose also that $f \in H^s(\Omega)$, $\widetilde{\Delta} f \in H^{s-1}(\Omega)$. Then $\widetilde{\partial} f \in H^s_{loc}(\Omega)$ and $T^I \widetilde{\partial} f \in L^2(\Omega)$ for all $|I| \leq s$ and there is a constant $C_s = C_s(M_0, ||\widetilde{x}||_{H^s(\Omega)})$ so that with notation as in (2.3.17), the following inequality holds for any $V \subset\subset \Omega$:*

$$||\widetilde{\partial} f||_{H^s(V)} + ||\mathcal{T}^s \widetilde{\partial} f||_{L^2(\Omega)} \leq C_s\big(||\widetilde{\Delta} f||_{H^{s-1}(\Omega)} + (||\mathcal{T}\widetilde{x}||_{H^s(\Omega)} + ||\widetilde{x}||_{H^s(\Omega)})||f||_{H^s(\Omega)}\big). \tag{2.B.62}$$

Proof. We will follow the proof in [12]. Both of the above statements have essentially the same proof and so we will just prove the second one. For the case $s = 1$, we want to show:

$$\sum_{T\in\mathcal{T}} ||T\widetilde{\partial} f||_{L^2(\Omega)} \le C(M')\big(||\widetilde{\Delta} f||_{L^2(\Omega)} + ||\widetilde{\partial} f||_{L^2(\Omega)}\big). \tag{2.B.63}$$

We fix one of the open sets $U = U_\mu$ with $\mu \ge 1$ and write $F = f \circ \psi_\mu$. Then, arguing as in [12], to prove (2.B.63) it suffices to prove that for every $V \subset U$, with a constant independent of h,

$$||D_c^h \widetilde{\partial} F||_{L^2(V)} \le C\big(||\widetilde{\Delta} f||_{L^2(\Omega)} + ||\widetilde{\partial} f||_{L^2(\Omega)}\big), \quad \text{for} \quad c = 1, 2, \tag{2.B.64}$$

for D_c^h denoting the difference quotient in the direction of a unit vector e_c

$$D_c^h F(z) = \big(F(z + he_c) - F(z)\big)/h. \tag{2.B.65}$$

Let ρ denote a cutoff function which is 1 on V and zero outside of U, and set $v = -D_c^{-h}(\rho^2 D_c^h F)$. Note that $v \in H_0^1(U)$. Now we have:

$$\begin{aligned}\int_U \widetilde{\Delta} F v &= -\int_U \delta^{ij}(\widetilde{\partial}_i F)\widetilde{\partial}_j\{D_c^{-h}(\rho^2 D_c^h F)\} \\ &= \int_U \delta^{ij}\underbrace{(\widetilde{\partial}_i F)D_c^{-h}\widetilde{\partial}_j\{\rho^2 D_c^h F\}}_{I} - \int_U \delta^{ij}\underbrace{(\widetilde{\partial}_i F)(D_c^{-h}A^a{}_j)\partial_a\{\rho^2 D_c^h F\}}_{II}.\end{aligned} \tag{2.B.66}$$

Now,

$$\begin{aligned} I &= \int_U (D_c^h \widetilde{\partial}_i F)\widetilde{\partial}_j(\rho^2 D_c^h F) = \int_U \rho^2 \delta^{ij}(D_c^h \widetilde{\partial}_i F)(\widetilde{\partial}_j D_c^h F) + \int_U 2\delta^{ij}(D_c^h \widetilde{\partial}_i f)(D_c^h f)(\rho\widetilde{\partial}_j \rho) \\ &= \int_U \rho^2 \delta^{ij}(D_c^h \widetilde{\partial}_i F)(D_c^h \widetilde{\partial}_j F) + \int_U \delta^{ij}(D_c^h \widetilde{\partial}_i F)\{2\rho(\widetilde{\partial}_j \rho)(D_c^h F) - \rho^2(D_c^h A^a{}_j)\partial_a F)\}.\end{aligned} \tag{2.B.67}$$

The first term is:

$$\int_U \rho^2 |D_c^h \widetilde{\partial} F|^2. \tag{2.B.68}$$

The second term is bounded by:

$$C(M_0)||\rho D_c^h\widetilde{\partial}F||_{L^2(U)}\big(||D_c^hF||_{L^2(U)}+||D_c^hA||_{L^\infty(U)}||f||_{H^1}\big)$$

$$\leq C(M_0)||\rho D_c^h\widetilde{\partial}F||_{L^2(U)}||f||_{H^1(\Omega)}(1+||\partial^2\widetilde{x}||_{L^\infty}). \quad (2.B.69)$$

Next, writing $\partial_a D_c^h F = D_c^h\partial_a F = D_c^h(A^\ell{}_a\widetilde{\partial}_\ell F)$:

$$II = \int_U \delta^{ij}(\widetilde{\partial}_j F)(D_c^{-h}A^a{}_j)\{2\rho\partial_a\rho D_c^hF+\rho^2\partial_a D_c^hF\}$$

$$= \int_U \delta^{ij}(\widetilde{\partial}_j F)(D_c^{-h}A^a{}_j)\{2\rho\partial_a\rho D_c^hF+\rho^2A^\ell{}_aD_c^h\widetilde{\partial}_\ell F-\rho^2(D_c^hA^\ell{}_a)\widetilde{\partial}_\ell F\} \quad (2.B.70)$$

so we have:

$$|II| \leq C(M_0)||\partial^2\widetilde{x}||_{L^\infty(U)}||\widetilde{\partial}F||_{L^2(U)}\times$$

$$\big(||F||_{H^1(U)}+||\rho D_c^h\widetilde{\partial}F||_{L^2(U)}+||\partial^2 x||_{L^\infty(U)}||\widetilde{\partial}F||_{L^2(U)}\big). \quad (2.B.71)$$

Finally, we have:

$$\int_U |\widetilde{\Delta}F||v|\,dy \leq ||\widetilde{\Delta}F||_{L^2(U)}||D_c^{-h}(\rho^2D_c^hF)||_{L^2(U)}. \quad (2.B.72)$$

Using similar arguments to the above, we can show:

$$||D_c^{-h}(\rho^2D_c^hF)||_{L^2(U)} \leq C(M')\big(||\rho D_c^h\widetilde{\partial}F||_{L^2(U)}+||\partial^2\widetilde{x}||_{L^\infty(U)}||F||_{H^1}\big), \quad (2.B.73)$$

so that:

$$\int_U \rho^2 |D_c^h \widetilde{\partial} F|^2$$

$$\leq C(M_0)\big(||\rho D_c^h \widetilde{\partial} F||_{L^2(U)}\big\{(1+||\partial^2 x||_{L^\infty(U)})||F||_{H^1(U)} + ||g||_{L^2(U)}\big\} + ||f||^2_{H^1(U)}\big). \quad (2.B.74)$$

Absorbing this first factor into the left-hand side we have, for any h small enough:

$$\int_V |D_c^h \widetilde{\partial} F|^2 \leq \int_U \rho^2 |D_c^h \widetilde{\partial} F|^2 \leq C(M_0)\big((1+||\partial^2 \widetilde{x}||_{L^\infty(U)})^2 ||F||^2_{H^1(U)} + ||g||^2_{L^2(U)}\big), \quad (2.B.75)$$

which implies the $s = 1$ case of the theorem.

Now suppose that $T^J \widetilde{\partial} F \in L^2(\Omega)$ for all $|J| \leq s-1$. Fix a multi-index I with $|I| = s-1$ and write $F' = T^I F$. Note that $F' = 0$ on $\partial\Omega$ in the trace sense and also that:

$$||\partial_y F'||_{L^2(\Omega)} \leq C(M_0) \widetilde{\partial} F'||_{L^2(\Omega)} \leq C(M_0)\big(||T^I \widetilde{\partial} F||_{L^2(\Omega)} + ||[\widetilde{\partial}, \mathcal{T}^I]F||_{L^2(\Omega)}\big). \quad (2.B.76)$$

The commutator can be bounded using Lemma 2.D.4:

$$||[\widetilde{\partial}, T^I]F||_{L^2(\Omega)} \leq C(M_0, ||\widetilde{x}||_{H^s(\Omega)})\big(||\mathcal{T}\widetilde{x}||_{H^s(\Omega)} + ||\widetilde{x}||_{H^s(\Omega)}\big)||F||_{H^{s-1}(\Omega)}. \quad (2.B.77)$$

In particular this implies that $F' \in H^1_0(\Omega)$. We also have:

$$\widetilde{\Delta} F' = T^I \widetilde{\Delta} F + [T^I, \widetilde{\Delta}]F, \quad (2.B.78)$$

and

$$||[T^I, \widetilde{\Delta}]F||_{L^2(\Omega)} \leq C(M_0, ||\widetilde{x}||_{H^s(\Omega)})(||\mathcal{T}\widetilde{x}||_{H^s(\Omega)} + ||\widetilde{x}||_{H^s(\Omega)})||F||_{H^s(\Omega)}, \quad (2.B.79)$$

Therefore we have that $F' \in H^1_0$ is the weak solution to the problem (2.B.78) and $\widetilde{\Delta} F' \in L^2(\Omega)$, so by the $|I| = 1$ case we have $\mathcal{T}\widetilde{\partial} F' \in L^2(\Omega)$ and:

$$||T\widetilde{\partial} F'||_{L^2(\Omega)} \leq C(M_0)\big(||\widetilde{\Delta} F'||_{L^2(\Omega)} + ||\widetilde{\partial} F'||_{L^2(\Omega)}\big). \quad (2.B.80)$$

We write:

$$T\widetilde{\partial}_i F' = T(\widetilde{\partial}_i \mathcal{T}^I F) = T T^I \widetilde{\partial}_i F + (T T^I A^a{}_i)\partial_a F + R, \tag{2.B.81}$$

where the L^2 norm of R is bounded by the right side of (2.B.62). Combining this with (2.B.80) gives (2.B.62). To prove the first estimate in (2.B.62) we argue in the same way, but we also prove (2.B.64) also for $c\!=\!3$. □

Proof of Proposition 2.B.3

We will need a few preliminary results. First, we fix a function d with $d = 0$ on $\partial\Omega$, $d < 0$ in Ω and $|\nabla d| > 0$ everywhere, so that the normal can be written as:

$$N_i = \widetilde{\partial}_i d \,/\, |\widetilde{\partial} d| = A^a{}_i \partial_a d \,/\, |\widetilde{\partial} d|, \qquad \text{where} \quad |\widetilde{\partial} d|^2 = \delta^{ij} \widetilde{\partial}_i d\, \widetilde{\partial}_j d \;= \widetilde{g}^{ab} \partial_a d\, \partial_b d. \tag{2.B.82}$$

By (2.D.2) and Lemma 2.D.1, this implies the estimates:

$$||N||_{C^\ell(\partial\Omega)} \le C(M')||\widetilde{x}||_{C^{\ell+1}(\partial\Omega)} \le C(M')||\widetilde{x}||_{H^{\ell+4}(\Omega)}, \tag{2.B.83}$$

$$||N||_{H^\ell(\partial\Omega)} \le C(M')||\widetilde{x}||_{H^{\ell+1}(\partial\Omega)}. \tag{2.B.84}$$

where in the first inequality we used Sobolev embedding on $\partial\Omega$ and the trace inequality (2.A.41). Recalling the definition $\gamma_{ij} = \delta_{ij} - N_i N_j$, there are similar estimates for derivatives of γ.

The basic result we need is the following consequence of Green's formula:

Lemma 2.B.8. *If α is a vector field then:*

$$||\widetilde{\partial}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)} = ||\operatorname{div}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)} + \frac{1}{2}||\operatorname{curl}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)} + \int_{\partial\widetilde{\mathcal{D}}_t} \Big(\alpha^j(\gamma_j^k \widetilde{\partial}_k \alpha_i)N^i - \alpha_i(\gamma_j^k \widetilde{\partial}_k \alpha^j)N^i\Big). \tag{2.B.85}$$

Proof. Integrating by parts:

$$||\widetilde{\partial}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)} = -\int_{\widetilde{\mathcal{D}}_t} \delta^{ij}\alpha_i\widetilde{\Delta}\alpha_j + \int_{\partial\widetilde{\mathcal{D}}_t} \delta^{ij}\alpha_i N^k\widetilde{\partial}_k\alpha_j. \tag{2.B.86}$$

We insert the identity:

$$\Delta\alpha_j = \delta^{k\ell}\widetilde{\partial}_k(\widetilde{\partial}_\ell\alpha_j) = \delta^{k\ell}\widetilde{\partial}_k\big(\widetilde{\partial}_j\alpha_\ell + \operatorname{curl}\alpha_{\ell j}\big) = \widetilde{\partial}_j \operatorname{div}\alpha + \delta^{k\ell}\widetilde{\partial}_k \operatorname{curl}\alpha_{\ell j}, \tag{2.B.87}$$

into the first term in (2.B.86) and integrate by parts again:

$$\int_{\widetilde{\mathcal{D}}_t} \delta^{ij}\alpha_i\widetilde{\Delta}\alpha_j = \int_{\partial\widetilde{\mathcal{D}}_t} N^i\alpha_i \operatorname{div}\alpha + \delta^{ij}N^\ell\alpha_i \operatorname{curl}\alpha_{\ell j}dS$$

$$- \int_{\widetilde{\mathcal{D}}_t} (\operatorname{div}\alpha)^2 + \delta^{k\ell}\delta^{ij}\widetilde{\partial}_k\alpha_i \operatorname{curl}\alpha_{\ell j}. \tag{2.B.88}$$

Note that by the antisymmetry of curl:

$$\delta^{k\ell}\delta^{ij}\widetilde{\partial}_k\alpha_i \operatorname{curl}\alpha_{\ell j} = \frac{1}{2}\delta^{k\ell}\delta^{ij}(\widetilde{\partial}_k\alpha_i + \widetilde{\partial}_k\alpha_i) \operatorname{curl}\alpha_{\ell j} + \frac{1}{2}\delta^{k\ell}\delta^{ij}(\widetilde{\partial}_k\alpha_i - \widetilde{\partial}_k\alpha_i) \operatorname{curl}\alpha_{\ell j}$$

$$= \frac{1}{2}\delta^{k\ell}\delta^{ij} \operatorname{curl}\alpha_{ki} \operatorname{curl}\alpha_{\ell j}, \tag{2.B.89}$$

so (2.B.86) becomes:

$$||\widetilde{\partial}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)} = ||\operatorname{div}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)} + \frac{1}{2}||\operatorname{curl}\alpha||^2_{L^2(\widetilde{\mathcal{D}}_t)}$$

$$+ \int_{\partial\widetilde{\mathcal{D}}_t} N^k\alpha^j\widetilde{\partial}_k\alpha_j - N^i\alpha_i \operatorname{div}\alpha - N^\ell\alpha^j \operatorname{curl}\alpha_{\ell j}. \tag{2.B.90}$$

Here:

$$N^k\alpha^j\widetilde{\partial}_k\alpha_j - N^i\alpha_i \operatorname{div}\alpha - N^\ell\alpha^j \operatorname{curl}\alpha_{\ell j} = N^k\alpha^j\widetilde{\partial}_j\alpha_k - N^i\alpha_i \operatorname{div}\alpha$$

$$= N^k\alpha_\ell N^\ell N^j\widetilde{\partial}_j\alpha_k + N^k\alpha_\ell\gamma^{\ell j}\widetilde{\partial}_j\alpha_k - N^i\alpha_i(N^kN^\ell + \gamma^{\ell k})\widetilde{\partial}_k\alpha_\ell = N^k\alpha_\ell\gamma^{\ell j}\widetilde{\partial}_j\alpha_k - N^i\alpha_i\gamma^{\ell k}\widetilde{\partial}_k\alpha_\ell.$$

□

Lemma 2.B.9. *There is a constant C_1 depending on $M'$and $||\widetilde{x}||_{H^5(\Omega)}$ so that if α is a vector field on Ω then*

$$||\alpha||^2_{H^1(\Omega)} \leq C_1\Big(||\operatorname{div}\alpha||^2_{L^2(\Omega)} + ||\operatorname{curl}\alpha||^2_{L^2(\Omega)}$$
$$+ \sum_{\mu=1}^N \int_{\partial\Omega}\big(\langle\partial_\theta\rangle_\mu^{1/2}\alpha^i\big)\big(\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j\big)N_iN_j dS + ||\alpha||^2_{L^2(\partial\Omega)} + ||\alpha||^2_{L^2(\Omega)}\Big), \quad (2.B.91)$$

$$||\alpha||^2_{H^1(\Omega)} \leq C_1\Big(||\operatorname{div}\alpha||^2_{L^2(\Omega)} + ||\operatorname{curl}\alpha||^2_{L^2(\Omega)}$$
$$+ \sum_{\mu=1}^N \int_{\partial\Omega}\big(\langle\partial_\theta\rangle_\mu^{1/2}\alpha^i\big)\big(\langle\partial_\theta\rangle_\mu^{1/2}\alpha^j\big)\gamma_{ij} dS + ||\alpha||^2_{L^2(\partial\Omega)} + ||\alpha||^2_{L^2(\Omega)}\Big). \quad (2.B.92)$$

Proof. These estimates follow from the fact that for any $\epsilon > 0$:

$$\Big|\int_{\partial\Omega}\alpha^j(\gamma_j^k\widetilde{\partial}_k\alpha_i)N^i - \alpha_i(\gamma_j^k\widetilde{\partial}_k\alpha^j)N^i\, dS\Big|$$
$$\leq C(M', ||\widetilde{x}||_{H^5(\Omega)})\Big(\frac{1}{\epsilon}\sum_{\mu=1}^N ||(\langle\partial_\theta\rangle^{1/2}\alpha)\cdot N||^2_{L^2(\Omega)}$$
$$+ \epsilon\sum_{\mu=1}^N ||(\langle\partial_\theta\rangle^{1/2}\alpha)\cdot\gamma||^2_{L^2(\partial\Omega)} + ||\alpha||^2_{L^2(\Omega)}\Big). \quad (2.B.93)$$

To see that this estimate implies (2.B.91), we use (2.B.84) and the trace inequality (2.A.41) to control the second term by $C(M')||\alpha||_{H^{1/2}(\partial\Omega)} \leq C(M')||\alpha||_{H^1(\Omega)}$, and then take ϵ sufficiently small. The estimate (2.B.92) follows by instead using the trace estimate on the first term and taking ϵ sufficiently large.

To prove (2.B.93), we write $\gamma_j^k\widetilde{\partial}_k\alpha^j = \gamma_j^k\widetilde{\partial}_k(\gamma_\ell^j\alpha^\ell) - \gamma_j^k(\widetilde{\partial}_k\gamma_\ell^j)\alpha^\ell - \gamma_j^k(\widetilde{\partial}_k N^j)N_\ell\alpha^\ell$ and the left-hand side as:

$$\int_{\partial\Omega} \alpha^j\gamma_j^k\widetilde{\partial}_k(\alpha_i N^i) - \gamma_j^k(\widetilde{\partial}_k(\gamma_\ell^j\alpha^\ell))\alpha_i N^i$$

$$+ \int_{\partial\Omega} \gamma_j^k(\widetilde{\partial}_k\gamma_\ell^j)\alpha^\ell\alpha_i N^i + \gamma_j^k(\widetilde{\partial}_k N^j)N_\ell\alpha^\ell\alpha_i N^i - \alpha^j\alpha_i\gamma_j^k\widetilde{\partial}_k N^i. \quad \text{(2.B.94)}$$

The second integral is bounded by the right-hand side of (2.B.93), by (2.B.84). The first integral is bounded by the right-hand side of (2.B.93) using the fractional product rules (2.A.7) and (2.A.8) - (2.A.10). □

Proof of Proposition 2.B.3. By the previous lemma we have the result for $\ell = 1$. Assume that we have the result for $\ell = 1, ..., m-1$. To prove it for $\ell = m$, we write $\partial_y^m\alpha = \partial_y^{m-1}\widetilde{\partial}\alpha + [\partial^{m-1}, \widetilde{\partial}]\alpha$. This second term can be bounded by the third term on the right-hand side of (2.B.28) (resp. (2.B.29)) by using Lemma 2.D.1 and arguing as in the proof of Proposition 2.B.2. To control the first term, we apply (2.B.16) and we need to control $||\widetilde{\partial}\,\mathrm{div}\,\alpha||_{H^{m-2}(\Omega)}$, $||\widetilde{\partial}\,\mathrm{curl}\,\alpha||_{H^{m-2}(\Omega)}$ and $||\mathcal{T}^J\widetilde{\partial}\alpha||_{L^2(\Omega)}$ for all multi-indices with $|J| = m-1$. Writing $\widetilde{\partial} = A\cdot\partial_y$ and arguing as above, the first two terms are bounded by the right-hand side of (2.B.28) (resp. (2.B.29)). It therefore just remains to control the third term. We commute T^J with $\widetilde{\partial}$, apply (2.D.5) and again argue as in the proof of Proposition 2.B.2. Applying (2.B.91) (resp. (2.B.92)) and repeating the same argument as above completes the proof of Proposition 2.B.3. □

Proof of Lemma 2.B.4. It suffices to prove the claim for $f \in C_c^\infty(\Omega)$ by an approximation argument. Integrating by parts twice and using that $\partial_a \widetilde{\partial}_i = \widetilde{\partial}_i \partial_a - (\partial_a A^c{}_i)\partial_c$, we have:

$$(\widetilde{\Delta} f, \Delta f)_{L^2(\Omega)} = \int_\Omega \delta^{ij}\delta^{ab}(\widetilde{\partial}_i \widetilde{\partial}_j f)(\partial_a \partial_b f) = \int_\Omega \delta^{ij}\delta^{ab}(\partial_a \widetilde{\partial}_j f)(\partial_b \widetilde{\partial}_i f)$$

$$+ \int_\Omega \delta^{ij}\delta^{ab}(\widetilde{\partial}_j f)(\partial_a A^c{}_j)(\partial_c \partial_b f) - \int_\Omega \delta^{ij}\delta^{ab}(\partial_a \widetilde{\partial}_j f)(\partial_b A^d{}_i)(\partial_d f). \quad (2.B.95)$$

This implies that:

$$(\widetilde{\Delta} f, \Delta f)_{L^2(\Omega)} \geq C(M)\big(||\widetilde{\partial} f||^2_{H^1(\Omega)} - ||\widetilde{\partial} f||_{H^1(\Omega)} ||f||_{H^1(\Omega)}\big), \quad (2.B.96)$$

and the result follows. $\square$

2.C Proofs of Elliptic estimates for the Newton potential

In this section we record the elliptic estimates that are needed to control ϕ in Section 2.7. We will use the convention in (2.7.2) for functions C_s, C_s', C_s'', C_s''' throughout this section.

2.C.1 Estimates for Section 2.7.1

Let $\widehat{\mathcal{D}}_t$ be the extended fluid domain (see Section 2.7) and $\widehat{\partial}$ be the associated spatial derivative.

Lemma 2.C.1. *Suppose $r \geq 5$. If $\widehat{\Delta} f = g$ in $\widehat{\mathcal{D}}_t$, then for $j \leq r-1$:*

$$||\widehat{\partial}\mathcal{T}^j \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r \Big(\sum_{k \leq j+1} ||\mathcal{T}^k \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} + \sum_{k \leq 2} ||\mathcal{T}^k g||_{L^6(\widehat{\mathcal{D}}_t)}$$

$$+ ||\mathcal{T}^j g||_{L^2(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)}\Big). \quad (2.C.1)$$

If $j \leq r-2$ then (2.C.1) holds without the L^6 norms, and if $j \leq r-1$ it holds without the L^∞ norm. In *addition,*

$$||\widehat{\partial}\mathcal{T}^r\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r\big(\sum_{k\leq r+1}||\mathcal{T}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}+||\mathcal{T}^r g||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$+||\mathcal{T}\widetilde{x}||_{H^r(\Omega)}\big[\sum_{k\leq 4}||\mathcal{T}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}+\sum_{k\leq 2}||\mathcal{T}^k g||_{L^6(\widehat{\mathcal{D}}_t)}+||g||_{L^\infty(\widehat{\mathcal{D}}_t)}\big]\big). \quad (2.C.2)$$

Moreover, for $0 \leq \ell \leq 2$*,*

$$||\widehat{\partial}\mathcal{T}^\ell\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)} \leq C_r\big(\sum_{k\leq \ell}||\mathcal{T}^k g||_{L^6(\widehat{\mathcal{D}}_t)}+\sum_{k\leq \ell+2}||\mathcal{T}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}\big), \quad (2.C.3)$$

as well as

$$||\widehat{\partial}^2 f||_{L^\infty(\widehat{\mathcal{D}}_t)} \leq C_0\big(||g||_{L^\infty(\widehat{\mathcal{D}}_t)}+\sum_{|J|\leq 1}||\widehat{\partial}^J\mathcal{T}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)}\big). \quad (2.C.4)$$

The above estimates also hold in the domain $\widetilde{\mathcal{D}}_t$ *with* $\widetilde{\partial}$ *instead of* $\widehat{\partial}$*.*

Proof. The estimate (2.C.4) follows from the pointwise estimate (2.5.5) and Sobolev embedding:

$$||\widehat{\partial}^2 f||_{L^\infty(\widehat{\mathcal{D}}_t)} \leq C_0\big(||g||_{L^\infty(\widehat{\mathcal{D}}_t)}+||\mathcal{T}\widehat{\partial}f||_{L^\infty(\widehat{\mathcal{D}}_t)}\big)$$

$$\leq C_0\big(||g||_{L^\infty(\widehat{\mathcal{D}}_t)}+\sum_{|J|\leq 1}||\widehat{\partial}^J\mathcal{T}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)}\big). \quad (2.C.5)$$

By (2.5.5) we also have:

$$||\widehat{\partial}\mathcal{T}^\ell\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)} \leq C_0\big(||\operatorname{div}\mathcal{T}^\ell\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)}+||\operatorname{curl}\mathcal{T}^\ell\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)}+||\mathcal{T}^{1+\ell}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)}\big)$$

$$\leq C_0\big(||\mathcal{T}^\ell g||_{L^6(\widehat{\mathcal{D}}_t)}+\sum_{k_1+k_2=\ell-1}||(\mathcal{T}^{1+k_1}\widehat{A})(\widehat{\partial}\mathcal{T}^{k_2}\widehat{\partial}f)||_{L^6(\widehat{\mathcal{D}}_t)}+||\mathcal{T}^{\ell+1}\widehat{\partial}f||_{H^1(\widehat{\mathcal{D}}_t)}\big), \quad (2.C.6)$$

where the sum is not there if $\ell=0$. Putting $\mathcal{T}^{1+k_1}\widehat{A}$ into L^∞ and using induction, this implies that for $\ell \leq 2$:

$$||\widehat{\partial}\mathcal{T}^{\ell}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)} \leq C_0\big(||\mathcal{T}^{\ell}g||_{L^6(\widehat{\mathcal{D}}_t)} + \textstyle\sum_{\ell'\leq\ell}||\mathcal{T}^{\ell'+1}\widehat{\partial}f||_{H^1(\widehat{\mathcal{D}}_t)}\big). \tag{2.C.7}$$

We now prove (2.C.1), which, combined with (2.C.7) will also prove (2.C.3). We proceed by induction: for $j=0$, (2.C.2) without the L^6 and L^∞ norms is a direct consequence of (2.5.5). Now suppose that (2.C.2) is known for $j = 0, 1, \cdots, m-1 \leq r-1$. Using the pointwise estimate (2.5.5) we have:

$$||\widehat{\partial}\mathcal{T}^{m}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_0\big(||\operatorname{div}\mathcal{T}^{m}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} + ||\operatorname{curl}\mathcal{T}^{m}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathcal{T}^{m+1}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}\big). \tag{2.C.8}$$

Here div and curl stand for the divergence and curl with respect to $\widehat{\partial}$. Since $\operatorname{div}\mathcal{T}^{m}\widehat{\partial}f = \mathcal{T}^m g + \sum(\mathcal{T}^k\widehat{A})\widehat{\partial}\mathcal{T}^{\ell}\widehat{\partial}f$, where $\widehat{A} = (\widehat{A}^a_{\ i})$ and the sum is over $k+\ell = m$ with $k \geq 1$, we have

$$\operatorname{div}\mathcal{T}^{m}\widehat{\partial}f = \mathcal{T}^m g + \textstyle\sum(\mathcal{T}^{k_1}\partial\widehat{x})\cdots(\mathcal{T}^{k_s}\partial\widehat{x})(\widehat{\partial}\mathcal{T}^{\ell}\widehat{\partial}f). \tag{2.C.9}$$

The above sum is over $k_1+\cdots+k_s+\ell = k+\ell = m, k \geq 1$, and ∂ denotes the Lagrangian spatial derivative ∂_y. This is because $\mathcal{T}^k\widehat{A}$ is a sum of terms of the form $(\mathcal{T}^{k_1}\partial\widehat{x})\cdots(\mathcal{T}^{k_s}\partial\widehat{x})$. Now, we need to control $\sum(\mathcal{T}^{k_1}\partial\widehat{x})\cdots(\mathcal{T}^{k_s}\partial\widehat{x})(\widehat{\partial}\mathcal{T}^{\ell}\widehat{\partial}f)$ in $L^2(\widehat{\mathcal{D}}_t)$. When $\ell\geq 3$, then $k_1,\cdots,k_s\leq r-3$, so all terms involving $\widehat{x}$ can be controlled in L^∞ by $||\widehat{x}||_{H^r(\Omega^{d_0})}$ and we control $||\widehat{\partial}\mathcal{T}^{\ell}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$ by the inductive assumption since $\ell\leq m-1$.

We now consider the case that at least one of $k_1,\cdots,k_s \geq r-2$ so that $\ell \leq 2$. Since $r \geq 5$, at most one of the k_j, say k_1, can be greater than or equal to $r-2$. If $k_1 = r-2$ or $k_1 = r-1$, then by Sobolev embedding we control $||\mathcal{T}^{k_1}\partial\widehat{x}||_{L^3(\Omega^{d_0})} \leq C||\mathcal{T}\widehat{x}||_{H^{(r-1,1/2)}(\Omega^{d_0})}$, and the other terms involving $\widehat{x}$ can be controlled in L^∞ and hence by $||\widehat{x}||_{H^{r-1}(\Omega^{d_0})}$. Using the estimate (2.C.7), the inductive assumption and Hölder's inequality $||f_1f_2||_{L^2}\leq||f_1||_{L^6}||f_2||_{L^3}$, we control the L^2 norm of right-hand side of (2.C.9) by the right-hand side of (2.C.1).

The only remaining case is when $k_1 = r$, and to deal with this we bound $\mathcal{T}^r\partial\widehat{x}$ in L^2 and use (2.C.4) to bound the L^∞ norm of the term involving f, which gives:

$$||\text{div}\mathcal{T}^m\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r\Big(||\mathcal{T}^m g||_{L^2(\widehat{\mathcal{D}}_t)} + \sum_{\ell=0,1,2}||\mathcal{T}^\ell g||_{L^6(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)}$$

$$+(||\mathcal{T}\widehat{x}||_{H^r(\Omega^{d_0})}+1)\sum_{k\leq m-1}||\widehat{\partial}\mathcal{T}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}\Big). \quad (2.C.10)$$

By the inductive assumption, $||\text{div}\mathcal{T}^m\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$ is controlled by the right-hand side of (2.C.2). A similar argument shows that $||\text{curl}\mathcal{T}^m\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$ is bounded by the right-hand side of (2.C.1) (resp. (2.C.2)) with $s=m$ but with $||\widetilde{x}||_{H^m(\Omega)}, ||\mathcal{T}\widetilde{x}||_{H^m(\Omega)}$ replaced by $||\widehat{x}||_{H^m(\Omega^{d_0})}, ||\mathcal{T}\widehat{x}||_{H^m(\Omega^{d_0})}$. Using (2.7.5) completes the proof. □

We also need the following estimate for the Newton potential:

Lemma 2.C.2. *If g is a smooth function supported in $\widehat{x}(t,\Omega^{d_0/2})$, then there is a constant C with:*

$$|\widehat{\partial}^s(g*\Phi)(x)| \leq C||g||_{L^2(\widehat{\mathcal{D}}_t)}, \qquad x \in \partial\widehat{\mathcal{D}}_t, \qquad s \geq 0. \quad (2.C.11)$$

Proof. Since there exists $c_0 > 0$ such that $d(\widehat{x}(t,\Omega^{d_0/2}), \partial\widehat{\mathcal{D}}_t) \geq c_0$, we have that $d(x,z) \geq c_0$ for each $z \in \text{supp}(g) \subset \widehat{x}(t,\Omega^{d_0/2})$, and so $\widehat{\partial}^s\Phi(x-\cdot) \in L^2(\widehat{x}(t,\Omega^{d_0/2}))$. Therefore,

$$|\widehat{\partial}^s(g*\Phi)| \leq ||g||_{L^2(\widehat{\mathcal{D}}_t)}||\widehat{\partial}^s\Phi(x-z)||_{L^2_z(\widehat{x}(t,\Omega^{d_0/2})} \leq C||g||_{L^2(\widehat{\mathcal{D}}_t)}.$$

□

Proof of Theorem 2.7.2. We proceed by induction. Write $f = g*\Phi$. When $j=0$ we have

$$||\widehat{\partial}f||^2_{L^2(\widehat{\mathcal{D}}_t)} = \int_{\widehat{\mathcal{D}}_t}\delta^{ij}(\widehat{\partial}_i f)\cdot(\widehat{\partial}_j f)\,dx = \int_{\partial\widehat{\mathcal{D}}_t}N^i(\widehat{\partial}_i f)f\,dS(x) - \int_{\widehat{\mathcal{D}}_t}gf\,dx. \quad (2.C.12)$$

By Lemma 2.C.2, the boundary integral in (2.C.12) is bounded by $C||g||^2_{L^2(\widehat{\mathcal{D}}_t)}$. The second term in (2.C.12) is bounded by $||g||_{L^2(\widehat{\mathcal{D}}_t)}||f||_{L^2(\widehat{\mathcal{D}}_t)}$, and by Young's inequality:

$$||f||_{L^2(\widehat{\mathcal{D}}_t)} = ||g*\Phi||_{L^2(\widehat{\mathcal{D}}_t)} \leq C||g||_{L^2(\widehat{\mathcal{D}}_t)}||\Phi||_{L^1(\widehat{\mathcal{D}}_t)} \leq C||g||_{L^2(\widehat{\mathcal{D}}_t)},$$

By (2.C.12), this implies:

$$||\widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C||g||_{L^2(\widehat{\mathcal{D}}_t)}. \tag{2.C.13}$$

Suppose that we now know that $||\mathcal{T}^j\widehat{\partial} g||_{L^2(\widetilde{\mathcal{D}}_t)}$ is bounded by the right-hand side of (2.7.2) for $j = 0, \cdots, m-1 \leq r-1$. To prove that it holds for $j = m$ as well, we integrate by parts:

$$\begin{aligned} ||\mathcal{T}^m\widehat{\partial} f||^2_{L^2(\widehat{\mathcal{D}}_t)} &= \int_{\widehat{\mathcal{D}}_t} (\mathcal{T}^m\widehat{\partial}_i f)(\mathcal{T}^m\widehat{\partial}^i f)\, dx \\ &= \int_{\widehat{\mathcal{D}}_t} \delta^{ij}\underbrace{(\mathcal{T}^m\widehat{\partial}_i f)\widehat{\partial}_j\mathcal{T}^m f dx}_{I} - \int_{\widehat{\mathcal{D}}_t}\delta^{ij}\underbrace{(\mathcal{T}^m\widehat{\partial}_i f)(\widehat{\partial}_j\mathcal{T}^m\widehat{x})\widehat{\partial} f dx}_{II} + \\ &\sum\int_{\widehat{\mathcal{D}}_t}\underbrace{(\mathcal{T}^m\widehat{\partial} f)(\partial\mathcal{T}^{\ell_1}\widehat{x})\cdots(\partial\mathcal{T}^{\ell_{s-1}}\widehat{x})\mathcal{T}^{\ell_s}\widehat{\partial} f dx}_{III}, \end{aligned} \tag{2.C.14}$$

where the sum is over $\ell_1 + \cdots + \ell_s = m$ and $\ell_1, \cdots, \ell_s \leq m-1$, $\ell_1 \geq 1$. To control III, we note that if $\ell_1, \cdots, \ell_{s-1} \leq r-3$, then

$$III \leq C(||\widetilde{x}||_{H^{r-1}(\Omega)})||\mathcal{T}^m\widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)}||\mathcal{T}^{\ell_s}\widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)},$$

and we control $||\mathcal{T}^{\ell_s}\widehat{\partial} f||_{L^2(\mathcal{D}_t)}$ by the inductive assumption. On the other hand, since $r \geq 5$, there can be at most one j with $\ell_j \geq r-2$ and without loss of generality it is ℓ_1 in which case $\ell_j \leq 2$ for $j = 2, 3, \cdots, s$. We then bound $III \leq C(||\widetilde{x}||_{H^{r-1}(\Omega)})||\mathcal{T}^m\widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)}||\partial\mathcal{T}^{\ell_1}\widehat{x}||_{L^3(\widehat{\mathcal{D}}_t)}||\mathcal{T}^{\ell_s}\widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)}$. By Sobolev embedding, $||\partial\mathcal{T}^{\ell_1}\widehat{x}||_{L^3(\widehat{\mathcal{D}}_t)} \leq C||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)}$, and $||\mathcal{T}^{\ell_s}\widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)}$ can be controlled using Lemma 2.C.1.

To control $I+II$, we integrate by parts and get

$$I+II=-\int_{\widehat{\mathcal{D}}_t}\delta^{ij}\underbrace{(\widehat{\partial}_i\mathcal{T}^m\widehat{\partial}_j f)\mathcal{T}^m f\,dx}_{I_1}+\int_{\widehat{\mathcal{D}}_t}\delta^{ij}\underbrace{(\widehat{\partial}_i\mathcal{T}^m\widehat{\partial}_j f)(\mathcal{T}^m\widehat{x}^k)\widehat{\partial}_k f\,dx}_{II_1}$$

$$+\int_{\widehat{\mathcal{D}}_t}\delta^{ij}\underbrace{(\mathcal{T}^m\widehat{\partial}_i f)(\mathcal{T}^m\widehat{x}^k)\widehat{\partial}_j\widehat{\partial}_k f\,dx}_{II_2}+\mathcal{B}, \quad (2.C.15)$$

where

$$\mathcal{B}=\int_{\partial\widehat{\mathcal{D}}_t}(N^i\mathcal{T}^m\widehat{\partial}_i f)\big(\mathcal{T}^m f-(\mathcal{T}^m\widehat{x}^k)(\widehat{\partial}_k f)\big). \quad (2.C.16)$$

To control II_1, we have:

$$\delta^{ij}\widehat{\partial}_i\mathcal{T}^m\widehat{\partial}_j f=\mathcal{T}^m\Delta f+(\partial\mathcal{T}^m\widehat{x})(\widehat{\partial}^2 f)+\sum(\partial\mathcal{T}^{\ell_1}\widehat{x})\cdots(\partial\mathcal{T}^{\ell_{s-1}}\widehat{x})(\widehat{\partial}\mathcal{T}^{\ell_s}\widehat{\partial}f), \quad (2.C.17)$$

where the sum is over $\ell_1+\cdots+\ell_s=m$ and $\ell_1,\cdots,\ell_s\le m-1$. The terms in the sum can be controlled similarly to how we controlled the sum in (2.C.9). The two main terms that are left in II_1 are

$$\int_{\widehat{\mathcal{D}}_t}(\mathcal{T}^m g)(\mathcal{T}^m\widehat{x})(\widehat{\partial}f)\,dx+\int_{\widehat{\mathcal{D}}_t}(\partial\mathcal{T}^m\widehat{x})(\widehat{\partial}^2 f)(\mathcal{T}^m\widehat{x})(\widehat{\partial}f)\,dx. \quad (2.C.18)$$

To control the second term in (2.C.18), we commute one $\mathcal{T}$ to the outside which gives:

$$\int_{\widehat{\mathcal{D}}_t}\underbrace{(\mathcal{T}\partial\mathcal{T}^{m-1}\widehat{x})(\widehat{\partial}^2 f)(\mathcal{T}^m\widehat{x})(\widehat{\partial}f)\,dx}_{II_{11}}+\int_{\widehat{\mathcal{D}}_t}\underbrace{(\partial^2\widehat{x})(\partial\mathcal{T}^{m-1}\widehat{x})(\widehat{\partial}^2 f)(\mathcal{T}^m\widehat{x})(\widehat{\partial}f)\,dx}_{II_{12}}. \quad (2.C.19)$$

To control II_{11}, we integrate half a tangential derivative by parts using (2.A.7) and get:

$$II_{11}\le C||\partial\mathcal{T}^{m-1}\widehat{x}||_{H^{(0,1/2)}(\Omega)}||(\widehat{\partial}^2 f)(\mathcal{T}^m\widehat{x})(\widehat{\partial}f)||_{H^{(0,1/2)}(\widehat{\mathcal{D}}_t)}. \quad (2.C.20)$$

Using the fractional product rule (2.A.8), for each μ we have with $L^2 = L^2(\widehat{\mathcal{D}}_t)$

$$||\langle\partial_\theta\rangle_\mu^{1/2}\big((\widehat{\partial}^2 f)(\mathcal{T}^m\widehat{x})\widehat{\partial} f\big)||_{L^2} \leq C||(\widehat{\partial}^2 f)(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^m\widehat{x})\widehat{\partial} f||_{L^2}$$

$$+\,||\mathcal{T}^m\widehat{x}||_{L^2(\Omega)}\sum_{\ell\leq 2}||\mathcal{T}^\ell\big((\widehat{\partial}^2 f)\widehat{\partial} f\big)||_{L^2}. \quad (2.C.21)$$

The first term on the right hand side can be controlled by

$$||\mathcal{T}\widetilde{x}||_{H^{(m-1,1/2)}(\Omega)}||\widehat{\partial}^2 f||_{L^\infty(\widehat{\mathcal{D}}_t)}||\widehat{\partial} f||_{L^\infty(\widehat{\mathcal{D}}_t)}.$$

Using (2.C.4), the Sobolev inequality $||\widehat{\partial} f||_{L^\infty(\widehat{\mathcal{D}}_t)} \leq C\sum_{|I|\leq 1}||\widehat{\partial}^{I+1} f||_{L^6(\widehat{\mathcal{D}}_t)}$ and (2.C.3), we control this term. To control the second term in (2.C.21), we just show how to control $||(\mathcal{T}^\ell\widehat{\partial}^2 f)(\widehat{\partial} f)||_{L^2(\widehat{\mathcal{D}}_t)}$ for $\ell\leq 2$ since the remaining terms are similar. For $\ell\leq 2$ we have:

$$||\mathcal{T}^\ell\widehat{\partial}^2 f||_{L^6(\widehat{\mathcal{D}}_t)} \leq ||\widehat{\partial}\mathcal{T}^\ell\widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)} + \sum_{\ell\leq 2}\sum_{j_1+j_2=\ell, j_1\geq 1}||(\mathcal{T}^{j_1}\widehat{A})(\widehat{\partial}\mathcal{T}^{j_2}\widehat{\partial} f)||_{L^6(\widehat{\mathcal{D}}_t)}. \quad (2.C.22)$$

By (2.C.3) we control the first term here, and after bounding the term involving $\widehat{A}$ in L^∞ and using (2.C.3) again we also control the second term. To control the term II_{12} from (2.C.19), we have:

$$II_{12} \leq P(||\mathcal{T}\widetilde{x}||_{H^{r-1}(\Omega)})||\widehat{\partial}^2 f||_{L^\infty(\widehat{\mathcal{D}}_t)}||\widehat{\partial} f||_{L^\infty(\widehat{\mathcal{D}}_t)}, \quad (2.C.23)$$

and then use (2.C.4). To control the first term in (2.C.18) we use (2.A.7) and then bound:

$$||\langle\partial_\theta\rangle_\mu^{1/2}\big((\mathcal{T}^m\widehat{x})(\widehat{\partial} f)\big)||_{L^2} \leq C\big(||(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^m\widehat{x})(\widehat{\partial} f)||_{L^2}$$

$$+\,||\langle\partial_\theta\rangle_\mu^{1/2}\big((\mathcal{T}^m\widehat{x})(\widehat{\partial} f)\big)-(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^m\widehat{x})(\widehat{\partial} f)||_{L^2}\big), \quad (2.C.24)$$

where $L^2 = L^2(\widehat{\mathcal{D}}_t)$, and then:

$$||(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^m\widehat{x})(\widehat{\partial}f)||_{L^2(\widehat{\mathcal{D}}_t)} \leq ||\mathcal{T}\widetilde{x}||_{H^{(r-1,1,1/2)}(\Omega)}||\widehat{\partial}f||_{L^\infty(\widehat{\mathcal{D}}_t)}$$

$$\leq ||\mathcal{T}\widetilde{x}||_{H^{(r-1,1,1/2)}(\Omega)}\sum_{\ell\leq 1}||\widehat{\partial}^{\ell+1}f||_{L^6(\widehat{\mathcal{D}}_t)}, \quad (2.C.25)$$

and by (2.A.8),

$$||\langle\partial_\theta\rangle_\mu^{1/2}\big((\mathcal{T}^m\widehat{x})(\widehat{\partial}f)\big) - (\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^m x)(\widehat{\partial}f)||_{L^2(\widehat{\mathcal{D}}_t)} \leq C||\mathcal{T}^m\widetilde{x}||_{L^2(\Omega)}\sum_{\ell\leq 2}||\mathcal{T}^\ell\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}. \quad (2.C.26)$$

To control II_2 in (2.C.15), we have

$$II_2 \leq ||\mathcal{T}^m\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}||\mathcal{T}^m\widehat{x}||_{L^3(\mathcal{D}_t)}||\widehat{\partial}^2 f||_{L^6(\mathcal{D}_t)}, \quad (2.C.27)$$

and $||\widehat{\partial}^2 f||_{L^6(\widehat{\mathcal{D}}_t)}$ under control, using (2.C.3).

To control I_1, we substitute (2.C.17) into I_1 and get, to highest order:

$$\int_{\widehat{\mathcal{D}}_t}(\mathcal{T}^m g)(\mathcal{T}^m f) + \int_{\widehat{\mathcal{D}}_t}(\mathcal{T}\partial\mathcal{T}^{m-1}\widehat{x})(\widehat{\partial}^2 f)(\mathcal{T}^m f). \quad (2.C.28)$$

We write $\mathcal{T} = \mathcal{T}^a\partial_{y^a} = \mathcal{T}^a\widehat{A}_a^i\widehat{\partial}_i$, so that:

$$\mathcal{T}^m f = (\mathcal{T}^a\widehat{A}_a^i\widehat{\partial}_i)\mathcal{T}^{m-1}f = \mathcal{T}^a\widehat{A}_a^i\Big(\mathcal{T}^{m-1}\widehat{\partial}_i f + (\widehat{\partial}_i\mathcal{T}^{m-1}\widehat{x})(\widehat{\partial}f)$$

$$+\sum_{\ell_1+\cdots+\ell_s\leq m-2}(\partial\mathcal{T}^{\ell_1}\widehat{x})\cdots(\partial\mathcal{T}^{\ell_{s-1}}\widehat{x})\mathcal{T}^{\ell_s}\widehat{\partial}f\Big). \quad (2.C.29)$$

Substituting this into (2.C.28), to highest order the result is:

$$\int_{\widehat{\mathcal{D}}_t} (\mathcal{T}^m g)(\mathcal{T}^a \widehat{A}^i_a)(\mathcal{T}^{m-1}\widehat{\partial}_i f) + \int_{\widehat{\mathcal{D}}_t} (\mathcal{T}^m g)(\mathcal{T}^a \widehat{A}^i_a)(\widehat{\partial}_i \mathcal{T}^{m-1}\widehat{x})(\widehat{\partial} f)$$
$$+ \int_{\widehat{\mathcal{D}}_t} (\mathcal{T}\partial\mathcal{T}^{m-1}\widehat{x})(\widehat{\partial}^2 f)(\mathcal{T}^a \widehat{A}^i_a)(\mathcal{T}^{m-1}\widehat{\partial}_i f)$$
$$+ \int_{\widehat{\mathcal{D}}_t} (\mathcal{T}\partial\mathcal{T}^{m-1}\widehat{x})(\widehat{\partial}^2 f)(\mathcal{T}^a \widehat{A}^i_a)(\widehat{\partial}_i \mathcal{T}^{m-1}\widehat{x})(\widehat{\partial} f). \quad (2.C.30)$$

The first and third terms can be controlled after integrating $\mathcal{T}$ by parts and using Hölder's inequality. The other terms can be controlled after integrating half a tangential derivative by parts using (2.A.7) and (2.A.8).

Finally, to control the boundary term $\mathcal{B}$ in (2.C.16), we use Lemma 2.C.2:

$$\mathcal{B} = \int_{\partial\widehat{\mathcal{D}}_t} (N^i \mathcal{T}^r \widehat{\partial}_i f)\Big(\mathcal{T}^r f - (\mathcal{T}^r \widehat{x})(\widehat{\partial} f)\Big) \le C\Big(||g||^2_{L^2(\widehat{\mathcal{D}}_t)} + ||g||^2_{L^2(\widehat{\mathcal{D}}_t)} \int_{\partial\widehat{\mathcal{D}}_t} |\mathcal{T}^r \widehat{x}|\Big), \quad (2.C.31)$$

which is controlled by $C(||\mathcal{T}\widetilde{x}||_{H^{(r-1,0.5)}(\Omega)} + 1)||g||_{L^2(\widehat{\mathcal{D}}_t)}$ by the trace lemma (2.A.41) and Theorem 2.A.1. ☐

2.C.2 Estimates for Section 2.7.2

Let $\mathfrak{D}^r$ be the mixed tangential space and time derivative defined in Section 2.3.3. We have:

Lemma 2.C.3. *Suppose that $r \ge 5$. If $\widehat{\Delta} f = g$ in $\widehat{\mathcal{D}}_t$, then for $j \le r-1$:*

$$||\widehat{\partial}\mathfrak{D}^j \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} \le C_r'\Big(\textstyle\sum_{k=0}^{j+1} ||\mathfrak{D}^k \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} + \sum_{k\le 2} ||\mathfrak{D}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + ||\mathfrak{D}^j g||_{L^2(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)}\Big). \quad (2.C.32)$$

In addition, we have:

$$||\widehat{\partial}\mathfrak{D}^r\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq P\big(||\widetilde{x}||_{H^r(\Omega)}, \textstyle\sum_{k\leq r-1}||D_t^k S_\varepsilon V||_{H^{r-k}(\Omega)}\big)$$

$$\cdot\big(\textstyle\sum_{k=0}^{r+1}||\mathfrak{D}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathfrak{D}^r g||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathcal{T}\widetilde{x}||_{H^r(\Omega)}\big[\textstyle\sum_{k\leq 4}||\mathfrak{D}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$+\textstyle\sum_{k\leq 2}||\mathfrak{D}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)}\big]\big). \quad (2.C.33)$$

Moreover, for $0\leq \ell\leq 2$*,*

$$||\widehat{\partial}\mathfrak{D}^\ell\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)} \leq C_r'\big(\textstyle\sum_{k\leq\ell}||\mathfrak{D}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + \sum_{k\leq \ell+2}||\mathfrak{D}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}\big). \quad (2.C.34)$$

Proof. It suffices to prove

$$||\widehat{\partial}\mathfrak{D}^{r-1}D_t\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C\big(||\widehat{x}||_{H^r(\Omega^{d_0})}, \textstyle\sum_{k\leq r-1}||D_t^k\widehat{V}||_{H^{r-k}(\Omega^{d_0})}\big)\big(\sum_{k=0}^{r+1}||\mathfrak{D}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$+||\mathfrak{D}^r g||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathcal{T}\widetilde{x}||_{H^{(r-1,0.5)}(\Omega^{d_0})}\big[\textstyle\sum_{k\leq 4}||\mathfrak{D}^k\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} + \sum_{k\leq 2}||\mathfrak{D}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)}\big]\big). \quad (2.C.35)$$

because (2.C.34) will then follow from this estimate and Lemma 2.C.1. Suppose that (2.C.35) is known for $||\widehat{\partial}\mathfrak{D}^{r-1}D_t\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$ with $j = 1,\cdots, r-2$, then for $j = r-1$, we have:

$$||\widehat{\partial}\mathfrak{D}^{r-1}D_t\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq ||\operatorname{div}\mathfrak{D}^{r-1}D_t\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$+||\operatorname{curl}\mathfrak{D}^{r-1}D_t\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} + ||\mathcal{T}\mathfrak{D}^{r-1}D_t\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}. \quad (2.C.36)$$

Here div and curl stand for the divergence and curl with respect to $\widehat{\partial}$. We only need to control the div term, because the curl term can be treated similarly. Since $\operatorname{div}\mathfrak{D}^{r-1}D_t\widehat{\partial}f = \mathfrak{D}^{r-1}D_t g + \sum(\mathfrak{D}^k\widehat{A})(\widehat{\partial}\mathfrak{D}^\ell\widehat{\partial}f)$, where $\widehat{A} = (\widehat{A}^a_{\ i})$ and the sum is over $k+\ell = r$ such that $k\geq 1$, we have

$$\operatorname{div} \mathfrak{D}^{r-1} D_t \widehat{\partial} f = \mathfrak{D}^{r-1} D_t g + \sum (\mathfrak{D}^{k_1} \partial \widehat{x}) \cdots (\mathfrak{D}^{k_s} \partial \widehat{x}) (\widehat{\partial} \mathfrak{D}^{\ell} \widehat{\partial} f). \tag{2.C.37}$$

The above sum is over $k_1 + \cdots + k_s + \ell = k + \ell = r$, which needs to be controlled in $L^2(\widehat{\mathcal{D}}_t)$. If $\ell \geq 3$, then $k_1, \cdots, k_s \leq r-3$, and so all terms involving $\widehat{x}$ can then be controlled in L^∞ by either $||\widehat{x}||_{H^r(\Omega^{d_0})}$ or $\sum_{k \leq r-3} ||D_t^k \widehat{V}||_{H^{r-k}(\Omega^{d_0})}$. Furthermore, when at least one of $k_1, \cdots, k_s \geq r-2$, since $r \geq 5$, there is at most one term, say k_1, can be greater than or equal to $r-2$. If $k_1 = r-2$ or $k_1 = r-1$, we control $||\mathfrak{D}^{k_1} \partial \widehat{x}||_{L^3(\Omega^{d_0})}$ by either $||\mathcal{T} \widehat{x}||_{H^{(r-1,0.5)}(\Omega^{d_0})}$ or $\sum_{k \leq r-2} ||D_t^k \widehat{V}||_{H^{r-k}(\Omega^{d_0})}$, and other terms involving $\widehat{x}$ are of lower order. In addition to this, we control $\widehat{\partial} \mathfrak{D}^\ell \widehat{\partial} f$ for $\ell \leq 2$ in L^6 because by the pointwise inequality (2.5.5) we have:

$$\begin{aligned} ||\widehat{\partial} \mathfrak{D}^\ell \widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)} &\leq C(M) \big(|| \operatorname{div} \mathfrak{D}^\ell \widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)} + || \operatorname{curl} \mathfrak{D}^\ell \widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)} + ||\mathcal{T} \mathfrak{D}^\ell \widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)} \big) \\ &\leq C(M) \Big(||\mathfrak{D}^\ell g||_{L^6(\widehat{\mathcal{D}}_t)} + \sum_{\ell_1 + \ell_2 = \ell, \ell_1 \geq 1} ||(\mathfrak{D}^{\ell_1} \widehat{A})(\widehat{\partial} \mathfrak{D}^{\ell_2} \widehat{\partial} f)||_{L^6(\widehat{\mathcal{D}}_t)} + ||\mathcal{T} \mathfrak{D}^\ell \widehat{\partial} f||_{H^1(\widehat{\mathcal{D}}_t)} \Big), \end{aligned} \tag{2.C.38}$$

where the second term is not present if $\ell = 0$. The second and third terms can be bounded by the right-hand side of (2.C.34) by the inductive assumption. On the other hand, when $k_1 = r$, $\mathfrak{D}^{k_1}$ involves at least one D_t, and so we control $\mathfrak{D}^{k_1} \partial \widehat{x}$ in L^2 by $\sum_{k \leq r-1} ||D_t^k \widehat{V}||_{H^{r-k}(\Omega^{d_0})}$. We also control $\widehat{\partial}^2 f$ in L^∞, as in Lemma 2.C.1. □

Lemma 2.C.4. *Fix $r \geq 7$. If g is a smooth function such that $supp(g) \subset \widehat{x}(t, \Omega^{d_0/2})$, then:*

$$\begin{aligned} ||D_t^r (g * \Phi)||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r' \Big(&\sum_{k \leq r-1} ||\mathfrak{D}^k \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} + \sum_{k \leq r} ||\mathfrak{D}^k g||_{L^2(\widehat{\mathcal{D}}_t)} \\ &+ \sum_{k \leq 2} ||\mathfrak{D}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)} \Big). \end{aligned} \tag{2.C.39}$$

Proof. Since $\widehat{\Delta} f = g$ in $\widehat{\mathcal{D}}_t$, commuting D_t^r through this and get

$$\widehat{\Delta} D_t^r f = (D_t^r g) + [\widehat{\Delta}, D_t^r] f. \tag{2.C.40}$$

In addition, since $D_t = \partial_t + \widehat{V}^k \widehat{\partial}_k$ in $\widehat{\mathcal{D}}_t$, we have $[\widehat{\partial}, D_t] = \widehat{\partial}\widehat{V} \cdot \widehat{\partial}$, which can then be used to compute

$$\begin{aligned}
[\widehat{\Delta}, D_t^r] = &\sum\nolimits_{\ell_1+\ell_2=r-1} c_{\ell_1,\ell_2} (\widehat{\Delta} D_t^{\ell_1} \widehat{V}) \cdot \widehat{\partial} D_t^{\ell_2} + \sum\nolimits_{\ell_1+\ell_2=r-1} c_{\ell_1,\ell_2} (\widehat{\partial} D_t^{\ell_1} \widehat{V}) \cdot \widehat{\partial} D_t^{\ell_2} \widehat{\partial} \\
&+ \sum\nolimits_{\ell_1+\cdots+\ell_n=r-n+1,\, n\geq 3} d_{\ell_1,\cdots,\ell_n} (\widehat{\partial} D_t^{\ell_3} \widehat{V}) \cdots (\widehat{\partial} D_t^{\ell_n} \widehat{V}) \cdot (\widehat{\partial}^2 D_t^{\ell_1} \widehat{V}) \cdot D_t^{\ell_2} \widehat{\partial} \\
&+ \sum\nolimits_{\ell_1+\cdots+\ell_n=r-n+1,\, n\geq 3} e_{\ell_1,\cdots,\ell_n} (\widehat{\partial} D_t^{\ell_3} \widehat{V}) \cdots (\widehat{\partial} D_t^{\ell_n} \widehat{V}) \cdot (\widehat{\partial} D_t^{\ell_1} \widehat{V}) \cdot \widehat{\partial} D_t^{\ell_2} \widehat{\partial}.
\end{aligned} \tag{2.C.41}$$

Since $\widehat{x}(t,y) = x_0(y)$ in $\Omega^{d_0} \backslash \Omega^{d_0/2}$, $[\widehat{\Delta}, D_t^r] f$ is compactly supported in $\widehat{x}(t, \Omega^{d_0/2})$. Therefore, (2.C.40) yields:

$$D_t^r f = (D_t^r g) * \Phi + ([\widehat{\Delta}, D_t^r] f) * \Phi. \tag{2.C.42}$$

The first term on the right can be controlled by $C(\mathrm{Vol}(\widehat{\mathcal{D}}_t)) ||D_t^r g||_{L^2(\widehat{\mathcal{D}}_t)}$ using Young's inequality. In addition, by (2.C.41), to control the $L^2(\widehat{\mathcal{D}}_t)$ norm of the second term it suffices to consider:

$$||[(\widehat{\partial}^2 D_t^{\ell_1} \widehat{V}) \cdots (\widehat{\partial} D_t^{\ell_{n-1}} \widehat{V}) \cdot D_t^{\ell_n} \widehat{\partial} f] * \Phi||_{L^2(\widehat{\mathcal{D}}_t)} \qquad ||[(\widehat{\partial} D_t^{\ell_1} \widehat{V}) \cdots (\widehat{\partial} D_t^{\ell_{n-1}} \widehat{V}) \cdot \widehat{\partial} D_t^{\ell_n} \widehat{\partial} f] * \Phi||_{L^2(\widehat{\mathcal{D}}_t)}, \tag{2.C.43}$$

where $\ell_1 + \cdots + \ell_n = r + 1 - n$ and $n \geq 2$. For the first term in (2.C.43), when $\ell_n \geq 3$, we must have $\ell_j \leq r - 4$ for $j \leq n - 1$. In this case, we bound the $\widehat{V}$ terms in $L^\infty(\widehat{\mathcal{D}}_t)$ and then use the Sobolev lemma:

$$||[(\widehat{\partial}^2 D_t^{\ell_1}\widehat{V})\cdots(\widehat{\partial}D_t^{\ell_{n-1}}\widehat{V})\cdot D_t^{\ell_n}\widehat{\partial}f]*\Phi||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$\leq C||(\widehat{\partial}^2 D_t^{\ell_1}\widehat{V})\cdots(\widehat{\partial}D_t^{\ell_{n-1}}\widehat{V})\cdot D_t^{\ell_n}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r'||D_t^{\ell_n}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}. \quad (2.C.44)$$

When $\ell_n = 1,2$, the worst case scenario is when $n=2$ and $D_t^{r-1-\ell_n}$ falls on $\widehat{\partial}^2\widehat{V}$. In other words, we only need to control $||[(\widehat{\partial}^2 D_t^{r-1-\ell_n}\widehat{V})(D_t^{\ell_n}\widehat{\partial}f)]*\Phi||_{L^2(\widehat{\mathcal{D}}_t)}$. Writing

$$[(\widehat{\partial}^2 D_t^{r-1-\ell_n}\widehat{V})(D_t^{\ell_n}\widehat{\partial}f)]*\Phi = \widehat{\partial}[(\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})(D_t^{\ell_n}\widehat{\partial}f)]*\Phi - (\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})(\widehat{\partial}D_t^{\ell_n}\widehat{\partial}f)*\Phi$$

$$= [(\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})(D_t^{\ell_n}\widehat{\partial}f)]*(\widehat{\partial}\Phi) - (\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})(\widehat{\partial}D_t^{\ell_n}\widehat{\partial}f)*\Phi, \quad (2.C.45)$$

and using that $\widehat{\partial}\Phi$ and Φ belong to $L^1(\widehat{\mathcal{D}}_t)$, Young's inequality implies that

$$||[(\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})D_t^{\ell_n}\widehat{\partial}f]*\widehat{\partial}\Phi||_{L^2(\widehat{\mathcal{D}}_t)} + ||[(\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})\widehat{\partial}D_t^{\ell_n}\widehat{\partial}f]*\Phi||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$\lesssim \sum_{k\leq 1}||(\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})\widehat{\partial}^k D_t^{\ell_n}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}. \quad (2.C.46)$$

Next, to control the term on the right hand side, we have:

$$\sum_{k\leq 1}||(\widehat{\partial}D_t^{r-1-\ell_n}\widehat{V})\widehat{\partial}^k D_t^{\ell_n}\widehat{\partial}f||_{L^2(\widehat{\mathcal{D}}_t)}$$

$$\leq C_r'\sum_{k\leq 1}||\widehat{\partial}D_t^{\ell_n}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)} \leq C_r'\big(||\widehat{\partial}D_t^{\ell_n}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)} + \sum_{k\leq 1}||\widehat{\partial}^k D_t^{\ell_n}\widehat{\partial}f||_{L^6(\widehat{\mathcal{D}}_t)}\big), \quad (2.C.47)$$

which can be controlled using Lemma 2.C.3. When $\ell_n = 0$, the worst-case scenario is when $n=2$ and D_t^{r-1} falls on $\widehat{\partial}^2\widehat{V}$. In other words, we only need to control $||(\widehat{\partial}^2 D_t^{r-1}\widehat{V})(\widehat{\partial}f)||_{L^2(\widehat{\mathcal{D}}_t)}$. By a similar argument as above, we need to control $\sum_{k=1,2}||(\widehat{\partial}D_t^{r-1}\widehat{V})(\widehat{\partial}^k f)||_{L^2(\widehat{\mathcal{D}}_t)}$, and this requires the control of $||\widehat{\partial}^k f||_{L^\infty(\widehat{\mathcal{D}}_t)}$ for $k=1,2$. The case when $k=2$ is treated in Lemma 2.C.1, and when $k=1$, we have by Young's inequality:

$$||\widehat{\partial} f||_{L^\infty(\widehat{\mathcal{D}}_t)} \le ||g * (\widehat{\partial}\Phi)||_{L^\infty(\widehat{\mathcal{D}}_t)} \le C||g||_{L^\infty(\widehat{\mathcal{D}}_t)}. \tag{2.C.48}$$

To control the $L^2(\widehat{\mathcal{D}}_t)$ norm for the second product in (2.C.43), when $\ell_n = r-1$ and $n=2$, we write:

$$[(\widehat{\partial}\widehat{V})(\widehat{\partial} D_t^{r-1}\widehat{\partial} f)] * \Phi = [(\widehat{\partial}\widehat{V})(D_t^{r-1}\widehat{\partial} f)] * (\widehat{\partial}\Phi) - [(\widehat{\partial}^2\widehat{V})(D_t^{r-1}\widehat{\partial} f)] * \Phi, \tag{2.C.49}$$

whose $L^2(\widehat{\mathcal{D}}_t)$ norm can then be controlled by $C_r' \sum_{k\le r-1} ||D_t^k \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)}$ using Young's inequality and Sobolev's lemma. When $r-2 \ge \ell_n \ge 2$ (and so $\ell_j \le r-3$ for $j=1,\cdots,n-1$), we have:

$$||[(\widehat{\partial} D_t^{\ell_1}\widehat{V})\cdots(\widehat{\partial} D_t^{\ell_{n-1}}\widehat{V})\cdot(\widehat{\partial} D_t^{\ell_n}\widehat{\partial} f)] * \Phi||_{L^2(\widehat{\mathcal{D}}_t)} \le C_r'||\widehat{\partial} D_t^{\ell_n}\widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)}, \tag{2.C.50}$$

using Young's inequality and Sobolev's lemma. The right hand side is controlled by Lemma 2.C.3. If $\ell_n = 1$, it suffices to consider $||[(\widehat{\partial} D_t^{r-2}\widehat{V})\widehat{\partial} D_t \widehat{\partial} f]*\Phi||_{L^2(\widehat{\mathcal{D}}_t)}$, which is bounded by $C_r'||\widehat{\partial} D_t \widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)}$. When $\ell_n = 0$, we need to control $||(\widehat{\partial} D_t^{r-1}\widehat{V})\widehat{\partial}^2 f]*\Phi||_{L^2(\widehat{\mathcal{D}}_t)}$, which requires control of $||\widehat{\partial}^2 f||_{L^\infty(\widehat{\mathcal{D}}_t)}$ as in in Lemma 2.C.1. □

Lemma 2.C.5. *There is a constant C so that if g is smooth and supported in $\widehat{x}(t,\Omega^{d_0/2})$ and $f = g*\Phi$ then*

$$|\widehat{\partial}^s D_t^k (g*\Phi)(x)| \le C||D_t^k g||_{L^2(\widehat{\mathcal{D}}_t)}, \qquad x \in \partial\widehat{\mathcal{D}}_t \qquad k,s \ge 0. \tag{2.C.51}$$

Proof. We have $[\widehat{\Delta}, D_t^k] f(x) = 0$ when $x \in \partial\widehat{\mathcal{D}}_t$ since $\widehat{V} = 0$ near $\partial\widehat{\mathcal{D}}_t$,. Therefore, (2.C.42) yields $\widehat{\partial}^s D_t^k f(x) = (D_t^k g) * (\widehat{\partial}^s \Phi)(x)$ and so (2.C.51) follows from a similar argument as in the proof of Lemma 2.C.2. □

Proof of Theorem 2.7.5. It suffices to prove that for $j \le r-1$:

$$\begin{aligned}||\mathfrak{D}^j D_t \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} &\le C\big(||\widetilde{x}||_{H^r(\Omega)}, \textstyle\sum_{k\le r-1}||D_t^k V||_{H^{r-k}(\Omega)}\big)\\ &\cdot ||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)}\big(\textstyle\sum_{k\le r}||\mathfrak{D}^k g||_{L^2(\widehat{\mathcal{D}}_t)} + \textstyle\sum_{k\le 2}||\mathfrak{D}^k g||_{L^6(\widehat{\mathcal{D}}_t)} + ||g||_{L^\infty(\widehat{\mathcal{D}}_t)}\big). \quad (2.C.52)\end{aligned}$$

When $j = r-1$, we have:

$$\begin{aligned}||\mathfrak{D}^{r-1} D_t \widehat{\partial} f||^2_{L^2(\widehat{\mathcal{D}}_t)} &= \int_{\widehat{\mathcal{D}}_t} \underbrace{\delta^{ij}(\mathfrak{D}^{r-1} D_t \widehat{\partial}_i f)(\widehat{\partial}_j \mathfrak{D}^{r-1} D_t f)\,dx}_{I}\\ &+ \int_{\widehat{\mathcal{D}}_t} \underbrace{\delta^{ij}(\mathfrak{D}^{r-1} D_t \widehat{\partial}_i f)([\mathfrak{D}^{r-1} D_t, \widehat{\partial}_j] f)}_{II}. \quad (2.C.53)\end{aligned}$$

We then control II by applying Corollary 2.D.1. To control I, we integrate by parts and get:

$$-\int_{\widehat{\mathcal{D}}_t} \underbrace{\delta^{ij}(\widehat{\partial}_i \mathfrak{D}^{r-1} D_t \widehat{\partial}_j f)(\mathfrak{D}^{r-1} D_t f)\,dx}_{I_1} + \int_{\partial\widehat{\mathcal{D}}_t} \underbrace{(N^i \mathfrak{D}^{r-1} D_t \widehat{\partial}_i f)(\mathfrak{D}^{r-1} D_t f)}_{\mathcal{B}}. \quad (2.C.54)$$

The interior term I_1 is equal to $\int_{\widehat{\mathcal{D}}_t}(\mathfrak{D}^{r-1} D_t g)(\mathfrak{D}^{r-1} D_t f)$ to highest order. The error terms here are as in (2.C.37), and the L^2 norm of these terms contribute $||\mathcal{T}\widetilde{x}||_{H^{(r-1,1/2)}(\Omega)}$ in (2.C.52) using (2.A.7). When $\mathfrak{D}^{r-1} = D_t^r$, this term can be controlled by $||D_t^r g||_{L^2(\widehat{\mathcal{D}}_t)}||D_t^r f||_{L^2(\widehat{\mathcal{D}}_t)}$, and then we may bound $||D_t^r f||_{L^2(\widehat{\mathcal{D}}_t)}$ using Lemma 2.C.4. In addition, when $\mathfrak{D}^{r-1} = \mathcal{T}\mathfrak{D}^{r-2}$, we control I_1 by integrating $\mathcal{T}$ by parts, similar to the control of (2.C.28) in the proof of Theorem 2.7.2. Finally, we use Lemma 2.C.5 to control $\mathcal{B}$. □

2.C.3 Estimates for Section 2.7.3

Theorem 2.C.1. *If $r \geq 5$, then for each $\mu = 0, ..., N$ and $j \leq r-1$:*

$$||\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^j\widehat{\partial}(g*\Phi)||_{L^2(\widehat{\mathcal{D}}_t)} \leq P(||\widetilde{x}||_{H^r(\Omega)})\Big(||g||_{L^\infty(\widehat{\mathcal{D}}_t)}$$
$$+\sum_{k\leq 2}||\mathcal{T}^k g||_{L^6(\widehat{\mathcal{D}}_t)}+\sum_{k\leq r-1}||\mathcal{T}^k g||_{L^2(\widehat{\mathcal{D}}_t)}\Big). \quad (2.C.55)$$

Proof. Suppose that we know (2.C.55) holds for $0 \leq j \leq r-2$, when $j = r-1$, we have

$$||\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}f||^2_{L^2(\widehat{\mathcal{D}}_t)} = \int_{\widehat{\mathcal{D}}_t}(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}_i f)(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}^i f)\,dx$$

$$= \int_{\widehat{\mathcal{D}}_t}\underbrace{(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}_i f)\langle\partial_\theta\rangle_\mu^{1/2}(\widehat{\partial}^i\mathcal{T}^{r-1}f)\,dx}_{I} - \int_{\widehat{\mathcal{D}}_t}\underbrace{(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}_i f)\langle\partial_\theta\rangle_\mu^{1/2}\big((\widehat{\partial}^i\mathcal{T}^{r-1}\widehat{x})(\widehat{\partial}f)\big)\,dx}_{II}$$
$$+\sum\int_{\widehat{\mathcal{D}}_t}\underbrace{(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}f)\langle\partial_\theta\rangle_\mu^{1/2}\big((\partial\mathcal{T}^{\ell_1}\widehat{x})\cdots(\partial\mathcal{T}^{\ell_{s-1}}\widehat{x})(\mathcal{T}^{\ell_s}\widehat{\partial}f)\big)\,dx}_{III}, \quad (2.C.56)$$

where the sum of over $\ell_1+\cdots+\ell_s = r-1$, $\ell_1,\cdots,\ell_s \leq r-2$. Invoking (2.A.8) and writing $L^2 = L^2(\widehat{\mathcal{D}}_t)$:

$$II \leq \int_{\widehat{\mathcal{D}}_t}(\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}f)(\langle\partial_\theta\rangle_\mu^{1/2}\widehat{\partial}\mathcal{T}^{r-1}\widehat{x})(\widehat{\partial}f)dx$$
$$+ C||\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}f||_{L^2}||\widehat{\partial}\mathcal{T}^{r-1}f||_{L^2}\sum_{k\leq 2}||\mathcal{T}^k\widehat{\partial}f||_{L^2}. \quad (2.C.57)$$

The last term on the right hand side is of the correct form that we control, while the main term is controlled as the corresponding term (i.e., II) in the proof of Theorem 2.7.2 and a repeated use of (2.A.8). In addition,

$$I \leq \int_{\widehat{\mathcal{D}}_t} (\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial}_i f)(\widehat{\partial}^i \langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1} f)dx$$

$$+ C||\langle\partial_\theta\rangle_\mu^{1/2}\mathcal{T}^{r-1}\widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)}||\widehat{\partial}\mathcal{T}^{r-1}f||_{L^2(\widehat{\mathcal{D}}_t)}\sum_{k\leq 2}||\mathcal{T}^k\widehat{A}||_{L^2(\widehat{\mathcal{D}}_t)}. \quad (2.C.58)$$

The last term on the right hand side is of the form that we control, while the main term can be controlled similarly to how we controlled the corresponding term (i.e., I) in the proof of Theorem 2.7.2 after a repeated use of (2.A.8). Finally, we need to control the L^2 norm of $\sum \langle\partial_\theta\rangle_\mu^{1/2}((\partial\mathcal{T}^{\ell_1}\widehat{x})\cdots(\partial\mathcal{T}^{\ell_{s-1}}\widehat{x})(\mathcal{T}^{\ell_s}\widehat{\partial}f))$ in III. When $\ell_s \geq 3$, then $\ell_1,\cdots,\ell_{s-1}\leq r-4$, and so we let $\langle\partial_\theta\rangle_\mu^{1/2}$ fall on $\partial\mathcal{T}^{\ell_1}\widehat{x}$ by applying (2.A.8) and then control the terms involving $\widehat{x}$ in L^∞. Moreover, if at least one of $\ell_1,\cdots,\ell_{s-1}$, say ℓ_1, is greater than or equal to $r-3$, we let $\langle\partial_\theta\rangle_\mu^{1/2}$ falls on $\partial\mathcal{T}^{\ell_1}\widehat{x}$ by applying (2.A.8) and control this term in L^3, and so $\mathcal{T}^s\widehat{\partial}f$ is controlled in L^6. But this can then be treated using Sobolev embedding and then Lemma 2.C.1. □

2.C.4 Estimates for Section 2.7.4

Lemma 2.C.6. *Suppose that $r\geq 7$ and f_J satisfy $\widehat{\Delta}_J f_J = g_J$ for $J = I, II$. Then for $j\leq r-1$, we have:*

$$||\widehat{\partial}_I\mathfrak{D}^j\widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathfrak{D}^j\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})} \leq D_r\Big(\sum_{k\leq r}||\mathfrak{D}^k\widehat{\partial}_I f_I - \mathfrak{D}^k\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}$$

$$+ ||\mathfrak{D}^{r-1}(g_I - g_{II})||_{L^2(\Omega^{d_0})} + \sum_{k\leq 2}||\mathfrak{D}^k(g_I - g_{II})||_{L^6(\Omega^{d_0})} + ||g_I - g_{II}||_{L^\infty(\Omega^{d_0})}$$

$$+ \Big\{||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} + \sum_{k\leq r-2}||D_t^k(V_I - V_{II})||_{H^{r-k}(\Omega)}\Big\}$$

$$\cdot\Big(\sum_{k\leq r}||\mathfrak{D}^k\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})} + ||\mathfrak{D}^{r-1}g_{II}||_{L^2(\Omega^{d_0})} + \sum_{k\leq 2}||\mathfrak{D}^k g_{II}||_{L^6(\Omega^{d_0})} + ||g_{II}||_{L^\infty(\Omega^{d_0})}\Big)\Big). \quad (2.C.59)$$

where $D_r = D_r\big(||\widetilde{x}_I||_{H^r(\Omega)}, ||\widetilde{x}_{II}||_{H^r(\Omega)}, \sum_{k\le r-2}||D_t^k V_I||_{H^{r-k}(\Omega)}, \sum_{k\le r-2}||D_t^k V_{II}||_{H^{r-k}(\Omega)}\big)$. *For* $0\le \ell \le 2$, *we have:*

$$||\widehat{\partial}_I \mathfrak{D}^\ell \widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathfrak{D}^\ell \widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})} \le D_r\Big(\sum_{k\le \ell}||\mathfrak{D}^k(g_I - g_{II})||_{L^6(\Omega^{d_0})}$$

$$+ \sum_{k\le \ell+2}||\mathfrak{D}^k\widehat{\partial}_I f_I - \mathfrak{D}^k\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)}\sum_{k\le \ell}||\widehat{\partial}_{II}\mathfrak{D}^k\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}\Big), \tag{2.C.60}$$

as well as

$$||\widehat{\partial}_I^2 f_I - \widehat{\partial}_{II}^2 f_{II}||_{L^\infty(\Omega^{d_0})} \lesssim ||g_I - g_{II}||_{L^\infty(\Omega^{d_0})}$$

$$+ \sum_{\ell\le 1}||\widehat{\partial}_I^\ell \mathcal{T}\widehat{\partial}_I f_I - \widehat{\partial}_{II}^\ell \mathcal{T}\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})} + ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)}\sum_{\ell\le 1}||\widehat{\partial}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}. \tag{2.C.61}$$

Proof. For $j=0$ (2.C.59) follows from (2.B.12). Suppose that (2.C.59) hold for $j\le r-2$. When $j=r-1$ we have:

$$||\widehat{\partial}_I \mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}$$

$$\lesssim ||\operatorname{div}_I \mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \operatorname{div}_{II}\mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})} + ||\operatorname{curl}_I \mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \operatorname{curl}_{II}\mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}$$

$$+ ||\mathcal{T}\mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \mathcal{T}\mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})} + ||\widehat{x}_I - \widehat{x}_{II}||_{H^r(\Omega^{d_0})}||\widehat{\partial}_{II}\mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}. \tag{2.C.62}$$

It suffices to control the div term, since the curl term can be controlled similarly. We have:

$$\operatorname{div}_I \mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \operatorname{div}_{II}\mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}$$

$$= \mathfrak{D}^{r-1}(g_I - g_{II}) + \sum\Big((\mathfrak{D}^{k_1}\partial\widehat{x}_I)\cdots(\mathfrak{D}^{k_s}\partial\widehat{x}_I)\widehat{\partial}_I\mathfrak{D}^\ell\widehat{\partial}_I f_I - (\mathfrak{D}^{k_1}\partial\widehat{x}_{II})\cdots(\mathfrak{D}^{k_s}\partial\widehat{x}_{II})\widehat{\partial}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}\Big). \tag{2.C.63}$$

where the sum is over $k_1+\cdots+k_s+\ell=r-1$, $k_1\geq 1$. To control the sum in $L^2(\Omega^{d_0})$ we only need to consider

$$\mathcal{A}=(\mathfrak{D}^{k_1}\partial\widehat{x}_I)\cdots(\mathfrak{D}^{k_s}\partial\widehat{x}_I)(\widehat{\partial}_I\mathfrak{D}^{\ell}\widehat{\partial}_I f_I-\widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}), \tag{2.C.64}$$

$$\mathcal{B}=(\mathfrak{D}^{k_1}\partial\widehat{x}_I-\mathfrak{D}^{k_1}\partial\widehat{x}_{II})(\mathfrak{D}^{k_2}\partial\widehat{x}_{II})\cdots(\mathfrak{D}^{k_s}\partial\widehat{x}_{II})(\widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}). \tag{2.C.65}$$

Now, if $\ell\geq 2$, then $k_1,\cdots,k_s\leq r-3$, and so all terms involving $\widehat{x}$ can then be controlled in L^∞, i.e.,

$$||\mathcal{A}||_{L^2(\Omega^{d_0})}\leq D_r||\widehat{\partial}_I\mathfrak{D}^{\ell}\widehat{\partial}_I f_I-\widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}, \tag{2.C.66}$$

$$||\mathcal{B}||_{L^2(\Omega^{d_0})}\leq D_r||\mathfrak{D}^{k_1}\partial\widehat{x}_I-\mathfrak{D}^{k_1}\partial\widehat{x}_{II}||_{L^\infty(\Omega^{d_0})}||\widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}. \tag{2.C.67}$$

Moreover, since $r\geq 7$, there is at most one of $k_1,\cdots,k_s$, say k_1, that can be $\geq r-2$. If $k_1=r-2$, then

$$||\mathcal{A}||_{L^2(\Omega^{d_0})}\leq D_r||\mathfrak{D}^{k_1}\partial\widehat{x}_I||_{L^3(\Omega^{d_0})}||\widehat{\partial}_I\mathfrak{D}^{\ell}\widehat{\partial}_I f_I-\widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}, \tag{2.C.68}$$

$$||\mathcal{B}||_{L^2(\Omega^{d_0})}\leq D_r||\mathfrak{D}^{k_1}\partial\widehat{x}_I-\mathfrak{D}^{k_1}\partial\widehat{x}_{II}||_{L^3(\Omega^{d_0})}||\widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}, \tag{2.C.69}$$

and since $\ell \leq 2$, we have:

$$||\widehat{\partial}_I\mathfrak{D}^\ell\widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$$

$$\lesssim ||\operatorname{div}_I\mathfrak{D}^\ell\widehat{\partial}_I f_I - \operatorname{div}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})} + ||\operatorname{curl}_I\mathfrak{D}^\ell\widehat{\partial}_I f_I - \operatorname{curl}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$$

$$+ ||\mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_I f_I - \mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})} + ||\widehat{x}_I - \widehat{x}_{II}||_{H^r(\Omega^{d_0})}||\widehat{\partial}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$$

$$\lesssim ||\mathfrak{D}^\ell(g_I - g_{II})||_{L^6(\Omega^{d_0})} + \sum_{\ell_1+\ell_2=\ell,\ell_1\geq 1}\Big(||(\mathfrak{D}^{\ell_1}\widehat{A}_I)(\widehat{\partial}_I\mathfrak{D}^{\ell_2}\widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathfrak{D}^{\ell_2}\widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})}$$

$$+ ||(\mathfrak{D}^{\ell_1}[\widehat{A}_I - \widehat{A}_{II}])(\widehat{\partial}_{II}\mathfrak{D}^{\ell_2}\widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})}\Big) + ||\mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_I f_I - \mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{H^1(\Omega^{d_0})}$$

$$+ ||\widehat{x}_I - \widehat{x}_{II}||_{H^r(\Omega^{d_0})}||\widehat{\partial}_{II}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}, \quad (2.C.70)$$

where the sum is of lower order and

$$||\partial_y(\mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_I f_I - \mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II})||_{L^2(\Omega^{d_0})} \lesssim ||\widehat{\partial}_I\mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}$$

$$+ ||(\widehat{\partial}_{II} - \widehat{\partial}_I)\mathcal{T}\mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}, \quad (2.C.71)$$

which is of the form we control. Finally, if $k_1 = r-1$, we need to control $\widehat{\partial}_I^2 f_I - \widehat{\partial}_{II}^2 f_{II}$ in L^∞. We have:

$$||\widehat{\partial}_I^2 f_I - \widehat{\partial}_{II}^2 f_{II}||_{L^\infty(\Omega^{d_0})} \lesssim ||g_I - g_{II}||_{L^\infty(\Omega^{d_0})}$$

$$+ ||\mathcal{T}(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^\infty(\Omega^{d_0})} + ||\widehat{x}_I - \widehat{x}_{II}||_{H^r(\Omega^{d_0})}||\widehat{\partial}_{II}^2 f_{II}||_{L^\infty(\Omega^{d_0})}, \quad (2.C.72)$$

where $||\mathcal{T}(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^\infty(\Omega^{d_0})} \lesssim \sum_{\ell\leq 1}||\partial_y^\ell\mathcal{T}(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})}$, and this can be controlled as above. □

Lemma 2.C.7. *Let $f_J = (g_J * \Phi) \circ \widehat{x}_J$ for $J = I, I\!I$, where g_J are smooth functions supported in $\Omega^{d_0/2}$ satisfying $\mathfrak{D} g_J = 0$ in $\Omega^{d_0} \backslash \Omega$. Then:*

$$||f_I - f_{I\!I}||_{L^2(\Omega^{d_0})} \le D_r(||g_I - g_{I\!I}||_{L^2(\Omega^{d_0})} + ||\widetilde{x}_I - \widetilde{x}_{I\!I}||_{H^r(\Omega)}||g_{I\!I}||_{L^2(\Omega^{d_0})}), \tag{2.C.73}$$

and for $r \ge 7$, we have:

$$\begin{aligned} ||D_t^{r-1} f_I - D_t^{r-1} f_{I\!I}||_{L^2(\Omega^{d_0})} \le D_r \Big(& \textstyle\sum_{k \le r-1} ||\mathfrak{D}^k \widehat{\partial}_I f_I - \mathfrak{D}^k \widehat{\partial}_{I\!I} f_{I\!I}||_{L^2(\Omega^{d_0})} \\ & + \textstyle\sum_{k \le r-1} ||\mathfrak{D}^k (g_I - g_{I\!I})||_{L^2(\Omega^{d_0})} + \sum_{k \le 2} ||\mathfrak{D}^k (g_I - g_{I\!I})||_{L^6(\Omega^{d_0})} + ||g_I - g_{I\!I}||_{L^\infty(\Omega^{d_0})} \\ & + \big\{ ||\widetilde{x}_I - \widetilde{x}_{I\!I}||_{H^r(\Omega)} + \textstyle\sum_{k \le r-2} ||D_t^k (V_I - V_{I\!I})||_{H^{r-k}(\Omega)} \big\} \\ & \cdot \big(\textstyle\sum_{k \le r-1} ||\mathfrak{D}^k \widehat{\partial}_{I\!I} f_{I\!I}||_{L^2(\Omega^{d_0})} + \sum_{k \le r-1} ||\mathfrak{D}^k g_{I\!I}||_{L^2(\Omega^{d_0})} \\ & + \textstyle\sum_{k \le 2} ||\mathfrak{D}^k g_{I\!I}||_{L^6(\Omega^{d_0})} + ||g_{I\!I}||_{L^\infty(\Omega^{d_0})} \big) \Big). \end{aligned} \tag{2.C.74}$$

Proof. We prove (2.C.73) first. Writing $f_J = \int_{\Omega^{d_0}} g_J(t, y') \Phi(\widehat{x}_J(t, y) - \widehat{x}_J(t, y')) \widehat{\kappa}_J \, dy$, we have

$$\begin{aligned} f_I - f_{I\!I} = & \int_{\Omega^{d_0}} \underbrace{g_{I\!I}(t, y') \big(\Phi(\widehat{x}_I(t, y) - \widehat{x}_I(t, y')) - \Phi\big(\widehat{x}_{I\!I}(t, y) - \widehat{x}_{I\!I}(t, y')\big) \big) \widehat{\kappa}_I \, dy'}_{I_1} \\ & + \int_{\Omega^{d_0}} \underbrace{(g_I(t, y') - g_{I\!I}(t, y')) \Phi\big(\widehat{x}_I(t, y) - \widehat{x}_I(t, y')\big) \widehat{\kappa}_I dy'}_{I_2} \\ & + \int_{\Omega^{d_0}} \underbrace{g_{I\!I}(t, y') \Phi\big(\widehat{x}_{I\!I}(t, y) - \widehat{x}_{I\!I}(t, y')\big) (\widehat{\kappa}_I - \widehat{\kappa}_{I\!I}) dy'}_{I_3}. \end{aligned} \tag{2.C.75}$$

By Young's inequality, we have:

$$||I_2||_{L^2(\Omega^{d_0})} \le D_r ||g_I - g_{I\!I}||_{L^2(\Omega^{d_0})}, \qquad ||I_3||_{L^2(\Omega^{d_0})} \le D_r ||\widetilde{x}_I - \widetilde{x}_{I\!I}||_{H^r(\Omega)} ||g_{I\!I}||_{L^2(\Omega^{d_0})}. \tag{2.C.76}$$

To control I_1, we write

$$\left|\Phi(\widehat{x}_I(t,y)-\widehat{x}_I(t,y'))-\Phi(\widehat{x}_{II}(t,y)-\widehat{x}_{II}(t,y'))\right|$$

$$=\frac{1}{4\pi}\left|\frac{1}{|\widehat{x}_I(t,y)-\widehat{x}_I(t,y')|}-\frac{1}{|\widehat{x}_{II}(t,y)-\widehat{x}_{II}(t,y')|}\right|$$

$$\leq\frac{1}{4\pi}\frac{|\widehat{x}_{II}(t,y)-\widehat{x}_I(t,y)|+|\widehat{x}_{II}(t,y')-\widehat{x}_I(t,y')|}{|\widehat{x}_I(t,y)-\widehat{x}_I(t,y')||\widehat{x}_{II}(t,y)-\widehat{x}_{II}(t,y')|}. \quad (2.C.77)$$

Since this is in $L^1(\Omega^{d_0})$, we have $||I_1||_{L^2(\Omega^{d_0})}\leq D_r(||\widetilde{x}_I||_{H^r(\Omega)})||\widetilde{x}_I-\widetilde{x}_{II}||_{H^r(\Omega)}||g_{II}||_{L^2(\Omega^{d_0})}$ using Young's inequality. Now, for (2.C.74), we write:

$$D_t^{r-1}f_I=(D_t^{r-1}g_I)*\Phi\circ\widehat{x}_I+([\widehat{\Delta}_I,D_t^{r-1}]f_I)*\Phi\circ\widehat{x}_I, \quad (2.C.78)$$

$$D_t^{r-1}f_{II}=(D_t^{r-1}g_{II})*\Phi\circ\widehat{x}_{II}+([\widehat{\Delta}_{II},D_t^{r-1}]f_{II})*\Phi\circ\widehat{x}_{II}. \quad (2.C.79)$$

To control $||D_t^{r-1}f_I-D_t^{r-1}f_{II}||_{L^2(\Omega^{d_0})}$, we need bounds for $||(D_t^{r-1}g_I)*\Phi\circ\widehat{x}_I-(D_t^{r-1}g_{II})*\Phi\circ\widehat{x}_{II}||_{L^2(\Omega^{d_0})}$ and $||([\widehat{\Delta}_I,D_t^{r-1}]f_I)*\Phi\circ\widehat{x}_I-([\widehat{\Delta}_{II},D_t^{r-1}]f_{II})*\Phi\circ\widehat{x}_{II}||_{L^2(\Omega^{d_0})}$. As above, they are controlled by

$$D_r\{||D_t^{r-1}g_I-D_t^{r-1}g_{II}||_{L^2(\Omega^{d_0})}+||\widetilde{x}_I-\widetilde{x}_{II}||_{H^r(\Omega)}||D_t^{r-1}g_{II}||_{L^2(\Omega^{d_0})}\}, \quad (2.C.80)$$

$$D_r\{||[\widehat{\Delta}_I,D_t^{r-1}]f_I-[\widehat{\Delta}_{II},D_t^{r-1}]f_{II}||_{L^2(\Omega^{d_0})}+||\widetilde{x}_I-\widetilde{x}_{II}||_{H^r(\Omega)}||[\widehat{\Delta}_{II},D_t^{r-1}]f_{II}||_{L^2(\Omega^{d_0})}\}, \quad (2.C.81)$$

respectively, where $[\widehat{\Delta}_{II},D_t^{r-1}]f_{II}$ can be treated by adapting the proof for Lemma 2.C.4. Moreover, since for each $J=I,II$, $[\widehat{\Delta}_J,D_t^{r-1}]$ consists

$$(\widehat{\partial}_J^2D_t^{\ell_1}\widehat{V}_J)\cdots(\widehat{\partial}_JD_t^{\ell_{n-1}}\widehat{V}_J)\cdot(D_t^{\ell_n}\widehat{\partial}_J)\quad\text{and}\quad(\widehat{\partial}_JD_t^{\ell_1}\widehat{V}_J)\cdots(\widehat{\partial}_JD_t^{\ell_{n-1}}\widehat{V}_J)\cdot(\widehat{\partial}_JD_t^{\ell_n}\widehat{\partial}_J), \quad (2.C.82)$$

where $\ell_1+\cdots+\ell_n=r-n$, the control of $||[\widehat{\Delta}_I,D_t^{r-1}]f_I-[\widehat{\Delta}_{II},D_t^{r-1}]f_{II}||_{L^2(\Omega^{d_0})}$ requires bounding:

$$K_1 = ||(\widehat{\partial}_I^2 D_t^{\ell_1} \widehat{V}_I) \cdots (\widehat{\partial}_I D_t^{\ell_{n-1}} \widehat{V}_I) \cdot (D_t^{\ell_n} \widehat{\partial}_I f_I)$$

$$- (\widehat{\partial}_{II}^2 D_t^{\ell_1} \widehat{V}_{II}) \cdots (\widehat{\partial}_{II} D_t^{\ell_{n-1}} \widehat{V}_{II}) \cdot (D_t^{\ell_n} \widehat{\partial}_{II} f_{II})||_{L^2(\Omega^{d_0})}, \quad (2.C.83)$$

$$K_2 = ||(\widehat{\partial}_I D_t^{\ell_1} \widehat{V}_I) \cdots (\widehat{\partial}_I D_t^{\ell_{n-1}} \widehat{V}_I) \cdot (\widehat{\partial}_I D_t^{\ell_n} \widehat{\partial}_I f_I)$$

$$- (\widehat{\partial}_{II} D_t^{\ell_1} \widehat{V}_{II}) \cdots (\widehat{\partial}_{II} D_t^{\ell_{n-1}} \widehat{V}_{II}) \cdot (\widehat{\partial}_{II} D_t^{\ell_n} \widehat{\partial}_{II} f_{II})||_{L^2(\Omega^{d_0})}. \quad (2.C.84)$$

: It suffices to consider the case when $n = 2$ only. To control K_1, we have, writing $L^2 = L^2(\Omega^{d_0})$:

$$||(\widehat{\partial}_I^2 D_t^{\ell_1} \widehat{V}_I) D_t^{\ell_2} \widehat{\partial}_I f_I - (\widehat{\partial}_{II}^2 D_t^{\ell_1} \widehat{V}_{II}) D_t^{\ell_2} \widehat{\partial}_{II} f_{II}||_{L^2}$$

$$\leq ||(\widehat{\partial}_I^2 D_t^{\ell_1} \widehat{V}_I - \widehat{\partial}_{II}^2 D_t^{\ell_1} \widehat{V}_{II}) D_t^{\ell_2} \widehat{\partial}_{II} f_{II}||_{L^2} + ||(\widehat{\partial}_I^2 D_t^{\ell_1} \widehat{V}_I)(D_t^{\ell_2} \widehat{\partial}_I f_I - D_t^{\ell_2} \widehat{\partial}_{II} f_{II})||_{L^2}$$

$$\leq \underbrace{||\big((\widehat{\partial}_I^2 - \widehat{\partial}_{II}^2) D_t^{\ell_1} \widehat{V}_I\big) D_t^{\ell_2} \widehat{\partial}_{II} f_{II}||_{L^2}}_{K_{12}} + \underbrace{||\big(\widehat{\partial}_{II}^2 D_t^{\ell_1} (\widehat{V}_I - \widehat{V}_{II})\big) D_t^{\ell_2} \widehat{\partial}_{II} f_{II}||_{L^2}}_{K_{13}}$$

$$+ \underbrace{||(\widehat{\partial}_I^2 D_t^{\ell_1} \widehat{V}_I)(D_t^{\ell_2} \widehat{\partial}_I f_I - D_t^{\ell_2} \widehat{\partial}_{II} f_{II})||_{L^2}}_{K_{11}}. \quad (2.C.85)$$

When $\ell_1 \leq r-4$, we bound $\widehat{V}$ factors in L^∞ and use Sobolev's lemma. Then $K_{11} \leq D_r ||D_t^{\ell_2}(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^2}$, and $K_{12} \leq D_r ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} ||D_t^{\ell_2} \widehat{\partial}_{II} f_{II}||_{L^2}$, and $K_{13} \leq D_r ||D_t^{\ell_1}(V_1 - V_2)||_{H^{r-\ell_1}} ||D_t^{\ell_2} \widehat{\partial}_{II} f_{II}||_{L^2}$. When $\ell_1 = r-3$ (and $\ell_2 = 1$), we bound $\widehat{V}$ terms in $L^3(\Omega^{d_0})$ and use Sobolev's lemma. In this case, $K_{11} \leq D_r ||D_t(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})}$, and $K_{12} \leq D_r ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} ||D_t \widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$ and $K_{13} \leq D_r ||D_t^{\ell_1}(V_1 - V_2)||_{H^{r-3}(\Omega)} ||D_t \widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$. By Sobolev's lemma $||D_t(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})} \lesssim$

$||D_t(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{H^1(\Omega^{d_0})}$, and we have

$$||\partial_y D_t(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^2(\Omega^{d_0})} \le D_r\big(||\widehat{\partial}_I D_t \widehat{\partial}_I f_I - \widehat{\partial}_{II} D_t \widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})} + ||(\widehat{\partial}_{II} - \widehat{\partial}_I) D_t \widehat{\partial}_{II} f_{II}||_{L^2(\Omega^{d_0})}\big),$$

which can be controlled by the right hand side of (2.C.74) using Lemma 2.C.3, and $||D_t \widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$ can be treated in a similar way. When $\ell_1 = r-2$ (and $\ell_2 = 0$), we bound $\widehat{V}$ terms in $L^2(\Omega^{d_0})$, so we need to control $||\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II}||_{L^\infty(\Omega^{d_0})}$ and $||\widehat{\partial}_{II} f_{II}||_{L^\infty(\Omega^{d_0})}$. By Sobolev's lemma, $||\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II}||_{L^\infty(\Omega^{d_0})} \lesssim \sum_{\ell \le 1} ||\partial_y^\ell(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})}$, where $||\partial_y(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})} \le ||\widehat{\partial}_I^2 f_I - \widehat{\partial}_{II}^2 f_{II}||_{L^6(\Omega^{d_0})} + ||(\widehat{A}_{II} - \widehat{A}_I)\partial_y \widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$, which is of the form that we control thanks to Lemma 2.C.3, and $||\widehat{\partial}_{II} f_{II}||_{L^\infty(\Omega^{d_0})}$ can be treated in a similar fashion. Finally, we control K_2 by adapting a similar argument as above. □

Lemma 2.C.8. *Let g_J, $J = I, II$ be smooth functions supported in $\Omega^{d_0/2}$ and $y \in \partial\Omega^{d_0}$. Then:*

$$|\partial_y \mathfrak{D}^k(f_I(t,y) - f_{II}(t,y))| \lesssim ||\mathfrak{D}^k(g_I - g_{II})||_{L^2(\Omega^{d_0})}$$

$$+ ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} ||\mathfrak{D}^k g_{II}||_{L^2(\Omega^{d_0})}, \quad k \ge 0. \quad (2.C.86)$$

Proof. Since $\widehat{x}(t,y) = x_0(y)$ and $\widehat{V}(t,y) = 0$ when $y \in \partial\Omega^{d_0}$, we have that $[\widetilde{\Delta}_I, \mathfrak{D}^k] f_I(y) = [\widetilde{\Delta}_{II}, \mathfrak{D}^k] f_{II}(y) = 0$. Therefore, $\partial_y \mathfrak{D}^k f_I(t,y) = (\mathfrak{D}^k g_I) * (\partial_y \Phi)(t,y)$ and $\partial_y \mathfrak{D}^k f_{II}(t,y) = (\mathfrak{D}^k g_{II}) * (\partial_y \Phi)(t,y)$, and the control of $|\partial_y \mathfrak{D}^k(f_I(t,y) - f_{II}(t,y))|$ follows from a similar argument that is used to control (2.C.75) since $\partial_y \Phi(\widehat{x}(t,y) - \widehat{x}(t,y'))$ is away from its singularity when $y' \in \Omega^{d_0/2}$ and $y \in \partial\Omega^{d_0}$. □

Theorem 2.C.2. *With the same assumptions as in Lemma 2.C.7, if $r \ge 7$, we have with $L^p =$*

$L^p(\Omega^{d_0})$:

$$\sum_{k\le r-1}||\mathfrak{D}^k\widehat{\partial}_I f_I - \mathfrak{D}^k\widehat{\partial}_{II} f_{II}||_{L^2} \le D_r\Big(\sum_{k\le r-1}||\mathfrak{D}^k(g_I - g_{II})||_{L^2}$$

$$+\sum_{k\le 2}||\mathfrak{D}^k(g_I-g_{II})||_{L^6} + ||g_I - g_{II}||_{L^\infty} \; + \big(||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} + \sum_{k\le r-2}||D_t^k(V_I - V_{II})||_{H^{r-k}(\Omega)}\big)$$

$$\times\big(\sum_{k\le r-1}||\mathfrak{D}^k g_{II}||_{L^2} + \sum_{k\le 2}||\mathfrak{D}^k g_{II}||_{L^6} + ||g_{II}||_{L^\infty}\big)\Big). \quad (2.C.87)$$

Proof. When $k = 0$, this is done as in the proof of Lemma 2.B.45. However, one needs to estimate $||f_I - f_{II}||_{L^2}$ directly without using Poincaré's inequality, which has been done in Lemma 2.C.7. Next, suppose that (2.C.87) is known for $k = 0, \cdots, r-2$. When $k = r-1$, we have:

$$||\mathfrak{D}^{r-1}\widehat{\partial}_I f_I - \mathfrak{D}^{r-1}\widehat{\partial}_{II} f_{II}||^2_{L^2(\Omega^{d_0})}$$

$$= \int_{\Omega^{d_0}} \delta^{ij}\underbrace{\big(\mathfrak{D}^{r-1}\widehat{\partial}_{Ii} f_I - \mathfrak{D}^{r-1}\widehat{\partial}_{IIi} f_{II}\big)\big(\widehat{\partial}_{Ij}\mathfrak{D}^{r-1} f_I - \widehat{\partial}_{IIj}\mathfrak{D}^{r-1} f_{II}\big)\, dy}_{I}$$

$$+ \int_{\Omega^{d_0}} \delta^{ij}\big(\mathfrak{D}^{r-1}\widehat{\partial}_{Ii} f_I - \mathfrak{D}^{r-1}\widehat{\partial}_{IIi} f_{II}\big)\big([\mathfrak{D}^{r-1},\widehat{\partial}_{Ij}] f_I - [\mathfrak{D}^{r-1},\widehat{\partial}_{IIj}] f_{II}\big)\, dy. \quad (2.C.88)$$

The second term can be bounded using Lemma 2.D.3 together with the bounds for $||\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II}||_{L^\infty(\Omega^{d_0})}$ and $\sum_{\ell\le 2}||\mathfrak{D}^\ell\widehat{\partial}_I f_I - \mathfrak{D}^\ell\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}$. Here, $||\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II}||_{L^\infty(\Omega^{d_0})} \lesssim \sum_{\ell\le 1}||\partial_y^\ell(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})}$, where

$$||\partial_y(\widehat{\partial}_I f_I - \widehat{\partial}_{II} f_{II})||_{L^6(\Omega^{d_0})} \le ||\widehat{\partial}_I^2 f_I - \widehat{\partial}_{II}^2 f_{II}||_{L^6(\Omega^{d_0})} + ||(\widehat{A}_{II}{}^a{}_i - \widehat{A}_I{}^a{}_i)\partial_{y^a}\widehat{\partial}_{II} f_{II}||_{L^6(\Omega^{d_0})}, \quad (2.C.89)$$

which is of the form that we control by Lemma 2.C.6. In addition, for each $\ell \le 2$, we have with $L^p = L^p(\Omega^{d_0})$:

$$||\mathfrak{D}^{\ell}\widehat{\partial}_I f_I - \mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^6} \lesssim ||\partial_y(\mathfrak{D}^{\ell}\widehat{\partial}_I f_I - \mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II})||_{L^2}$$

$$\leq ||\widehat{\partial}_I\mathfrak{D}^{\ell}\widehat{\partial}_I f_I - \widehat{\partial}_{II}\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^2} + ||(\widehat{A}_{II}{}^a_i - \widehat{A}_I{}^a_i)\partial_a\mathfrak{D}^{\ell}\widehat{\partial}_{II} f_{II}||_{L^2}, \quad (2.C.90)$$

which is again of the form that we control by Lemma 2.C.6. To deal with the first term in (2.C.88), one writes $\mathfrak{D}^{r-1}\widehat{\partial}_{IIi} f_{II} = \mathfrak{D}^{r-1}\widehat{\partial}_{Ii} f_{II} + \mathfrak{D}^{r-1}[(\widehat{A}_{II}{}^a_i - \widehat{A}_I{}^a_i)\partial_a f_{II}]$ and $\widehat{\partial}_{IIi}\mathfrak{D}^{r-1} f_{II} = \widehat{\partial}_{Ii}\mathfrak{D}^{r-1} f_{II} + (\widehat{A}_{II}{}^a_i - \widehat{A}_I{}^a_i)\partial_a\mathfrak{D}^{r-1} f_{II}$, and

$$I = \int_{\Omega^{d_0}} \delta^{ij}\big(\widehat{\partial}_{Ii}\mathfrak{D}^{r-1}(f_I - f_{II})\big)\big(\mathfrak{D}^{r-1}\widehat{\partial}_{Ij}(f_I - f_{II})\big)dy$$

$$+ \int_{\Omega^{d_0}} \delta^{ij}\big(\widehat{\partial}_{Ii}\mathfrak{D}^{r-1}(f_I - f_{II})\big)\big(\mathfrak{D}^{r-1}\big[(\widehat{A}_{II}{}^a_j - \widehat{A}_I{}^a_j)\partial_a f_{II}\big]\big)dy$$

$$+ \int_{\Omega^{d_0}} \delta^{ij}\big((\widehat{A}_{II}{}^a_i - \widehat{A}_I{}^a_i)\partial_a\mathfrak{D}^{r-1} f_{II}\big)\big(\mathfrak{D}^{r-1}\widehat{\partial}_{Ij}(f_I - f_{II})\big)dy$$

$$+ \int_{\Omega^{d_0}} \delta^{ij}\big((\widehat{A}_{II}{}^a_i - \widehat{A}_I{}^a_i)\partial_a\mathfrak{D}^{r-1} f_{II}\big)\big(\mathfrak{D}^{r-1}\big[(\widehat{A}_{II}{}^a_j - \widehat{A}_I{}^a_j)\partial_a f_{II}\big]\big)dy. \quad (2.C.91)$$

It is straightforward to control the last three terms, and the first term is equal to:

$$\int_{\Omega^{d_0}} \delta^{ij}\partial_a\big((\widehat{A}_I{}^a_i\mathfrak{D}^{r-1}(f_I - f_{II})\big)\mathfrak{D}^{r-1}\widehat{\partial}_{Ij}(f_I - f_{II})\big)dy$$

$$- \int_{\Omega^{d_0}} \delta^{ij}(\partial_a\widehat{A}_I{}^a_i)\big(\mathfrak{D}^{r-1}(f_I - f_{II})\big)\big(\mathfrak{D}^{r-1}\widehat{\partial}_{Ij}(f_I - f_{II})\big)\, dy. \quad (2.C.92)$$

Integrating by parts in the first term gives:

$$\int_{\Omega^{d_0}} \delta^{ij} \underbrace{\left(\mathfrak{D}^{r-1}(f_I - f_{II})\right)\widehat{\partial}_{Ii}\left(\mathfrak{D}^{r-1}\widehat{\partial}_{Ij}(f_I - f_{II})\right) dy}_{II}$$

$$+ \int_{\partial\Omega^{d_0}} \delta^{ij} \underbrace{N_a \widehat{A}_{I\,i}^{\;a}\left(\mathfrak{D}^{r-1}(f_I - f_{II})\right)\left(\mathfrak{D}^{r-1}\widehat{\partial}_{Ij}(f_I - f_{II})\right) dy}_{\mathcal{B}}. \quad (2.C.93)$$

Here, modulo controllable error terms, II is equal to $\int_{\Omega^{d_0}}(\mathfrak{D}^r f_I - \mathfrak{D}^r f_{II})(\mathfrak{D}^r g_I - \mathfrak{D}^r g_{II})\,dy$. When $\mathfrak{D}^{r-1}$ contains at least one $\mathcal{T}$, one can integrate this $\mathcal{T}$ by parts and control the resulting integral as what is done to the control of (2.C.28) in the proof of Theorem 2.7.2. When $\mathfrak{D}^{r-1} = D_t^{r-1}$, this is bounded by $||D_t^{r-1} f_I - D_t^{r-1} f_{II}||_{L^2(\Omega^{d_0})}||D_t^{r-1} g_I - D_t^{r-1} g_{II}||_{L^2(\Omega^{d_0})}$, where $||D_t^{r-1} f_I - D_t^{r-1} f_{II}||_{L^2(\Omega^{d_0})}$ can be controlled by Lemma 2.C.7. The second term in (2.C.92) can be controlled in a similar way. On the other hand, since $\widehat{A}_{I\,i}^{\;a} = \delta_i^a$ on $\partial\Omega^{d_0}$, $\mathcal{B}$ can be controlled appropriately using the Lemma 2.C.8. □

2.D Estimates for commutators and $\mathcal{F}$

In this section, we fix a vector field $V = V(t, y)$ on Ω. We let $x(t, y)$ denote the flow of $V(t, y)$, i.e. $D_t x = V$, $x|_{t=0} = x_0$, and let $\widetilde{x}(t, y)$ denote the tangentially smoothed flow, as in (2.4.1). We suppose that the mapping $y \mapsto \widetilde{x}(t, y)$ is invertible for each t, and we let $A^i{}_a$ and $A^a{}_i$ be the Jacobian matrix of $\widetilde{x}$ and its inverse, respectively, see (2.4.2). We will assume that $\widetilde{x}$ and V satisfy the bounds (2.5.1).

If $M^i{}_a$ is an invertible matrix with inverse N^a_i, we recall the formula for the derivatives of N^a_i:

$$DN^a_i = -N^b_i N^a_j (DM^j{}_b), \quad (2.D.1)$$

where here $D = D_t$ or $D = \partial_c$. When $M^i{}_a = A^i{}_a$, then this gives:

$$D_t A^a{}_i = -A^a{}_j\big(\widetilde{\partial}_i S_\varepsilon V^j\big), \qquad\qquad \partial_c A^a{}_i = -A^a{}_j\big(\widetilde{\partial}_i u^j_c\big). \tag{2.D.2}$$

Using these formulas it is straightforward to calculate the following commutators:

$$[D_t, \widetilde{\partial}_i] = -A^b{}_i A^a{}_j\big(\partial_b S_\varepsilon V^j\big)\partial_a = -\big(\widetilde{\partial}_i S_\varepsilon V^j\big)\widetilde{\partial}_j, \tag{2.D.3}$$

$$[\partial_c, \widetilde{\partial}_i] = -A^b{}_i A^a{}_j\big(\partial_c A^j{}_b\big)\partial_a = -\big(\widetilde{\partial}_i A^j{}_c\big)\widetilde{\partial}_j. \tag{2.D.4}$$

We will need estimates for higher order derivatives of $A^a{}_i$. As in section 2.3.3, given a set $\mathcal{U} = \{T_1, ..., T_N\}$ of vector fields, we write $\mathcal{U}^r = \mathcal{U} \times \cdots \times \mathcal{U}$ (r times) as well as $\mathcal{U}^r V : \Omega \to \mathbb{R}^{3N+3}$. The families of vector fields we will consider are $\mathcal{U} = \mathcal{T}$ (tangential derivatives, $\mathcal{U} = \mathfrak{D}$ (mixed tangential and time derivatives), $\mathcal{U} = \mathcal{D}$ (mixed full space and time derivatives), and $\mathcal{U} = \{\partial_y\}$. The point of the below estimate is just that derivatives of A behave like derivatives of $\partial_y \widetilde{x}$. This lemma is in fact essentially the same as Lemma 2.D.4 but it is convenient to note this estimate separately.

Lemma 2.D.1. *With notation as in Section 2.3.3, if $T^I \in \mathcal{U}^s$ where $\mathcal{U} = \mathcal{T}, \mathfrak{D}, \mathcal{D}$, or $\{\partial_y\}$, then:*

$$||T^I A^a{}_i||_{L^2} + ||T^I g^{ab}||_{L^2} \leq C(M)\big(||T^I \widetilde{x}||_{H^1} + P(||\mathcal{U}^{s-2}\widetilde{x}||_{H^2})\big) \tag{2.D.5}$$

We note that taking $\mathcal{U} = \{\partial_y\}$ and summing over all $T^I \in \mathcal{U}^s$ gives:

$$||A^a{}_i||_{H^s} + ||g^{ab}||_{H^s(\Omega)} \leq C(M)\big(||\widetilde{x}||_{H^{s+1}} + P(||\widetilde{x}||_{H^s})\big). \tag{2.D.6}$$

Proof. The estimates for g follow from the estimates for A and the definition $g^{ab} = \delta^{ij} A^a{}_i A^b{}_j$ so we just prove the estimates for A. For the sake of simplicity we will assume that all $T \in \mathcal{U}$ commutes with ∂_y; this is only not the case if $\mathcal{U} = \mathcal{T}$ and in that case the commutator is lower order and can be handled using similar arguments to the below. For $T^I \in \mathcal{U}^s$, repeatedly applying (2.D.2), we have:

$$T^I A^a_{\ i} = -A^b_{\ i} A^a_{\ j} T^I \partial_b \widetilde{x}^j - \sum \big(\widetilde{\partial} T^{I_1} \widetilde{x}\big) \cdots \big(\widetilde{\partial} T^{I_k} \widetilde{x}\big) \tag{2.D.7}$$

where the sum is taken over a collection of multi-indices $I_1, ..., I_k$ with $|I_1| + \cdots + |I_k| = s$ with $|I_j| \leq s-1$ for $j = 1, ..., k$. The first term is bounded by the first term on the right-hand side of (2.D.6). When $s \leq 3$, we bound the first $k-1$ factors in each summand in L^∞ by $C(M)$ and the remaining factor in L^2 by $||\mathcal{U}^{s-1}\widetilde{x}||_{H^1}$ and this is bounded by the right-hand side of (2.D.6) for all the values of $\mathcal{U}$ we are considering. We now assume that $s \geq 4$. If any index $|I_j| \leq \min(3, s-3)$, we use the Sobolev estimate (2.A.43) to bound $||\widetilde{\partial} T^{I_j} x||_{L^\infty} \leq C(M)||\partial_y T^{I_j} x||_{L^\infty} \leq C(M)||\mathcal{U}^{s-2} x||_{H^2}$. Therefore it suffices to deal with the case when at least one index $|I_j| \geq \max(4, s-4)$. There can be at most one such index because if there are $\ell \geq 2$ such terms then $4\ell \leq s$ so that $s \geq 8$ and that $\ell(s-4) \leq s$ so that $s \leq 4$. Since there is one such index and $|I_j| \leq s-1$ we bound the corresponding term in L^2 by $||\mathcal{U}^{s-1}\widetilde{x}||_{H^1}$ which completes the proof. □

Similarly, we have:

Lemma 2.D.2. *Define $\widetilde{x}_I, \widetilde{x}_{II}, A_I, A_{II}, \widetilde{g}_I, \widetilde{g}_{II}$ as in Appendix 2.B. With notation as in Lemma 2.D.1, if $T^I \in \mathcal{V}^s$:*

$$||T^I(A_{I\,i}^{\ a} - A_{II\,i}^{\ a})||_{L^2} + ||T^I(\widetilde{g}_I^{ab} - \widetilde{g}_{II}^{ab})||_{L^2} \leq \mathcal{D}_s ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^{\ell+1}}, \tag{2.D.8}$$

where

$$\mathcal{D}_s = \mathcal{D}_s(M, ||\mathcal{U}^{s-1}\widetilde{x}_I||_{H^2}, ||\mathcal{U}^{s-1}\widetilde{x}_{II}||_{H^2}). \tag{2.D.9}$$

Proof. Applying (2.D.2) to $A_{I\,a}^{\ i}$ and $A_{II\,a}^{\ i}$ generates two sums of the form (2.D.7). Subtracting these two sums and arguing as in the proof of the previous lemma gives (2.D.8). □

The next lemma will be used at several places. Recall the definitions of $\Omega^{d_0}, \widehat{\partial}_I, \widehat{\partial}_{II}$ from

Section 2.7.1.

Lemma 2.D.3. *Let with $r \geq 5$. Then there is a continuous function*

$$C_r = C_r\big(M', ||\widetilde{x}_I||_{H^r(\Omega)}, ||\widetilde{x}_{II}||_{H^r(\Omega)}, \textstyle\sum_{\ell \leq r-1} ||D_t^\ell V_I||_{H^{r-\ell}(\Omega)}, \sum_{\ell \leq r-1} ||D_t^\ell V_{II}||_{H^{r-\ell}(\Omega)}\big).$$

such that with $\mathfrak{D}^r$ the mixed space-time tangential derivatives defined in Section 2.3.3:

$$\begin{aligned}
&||[\mathfrak{D}^r, \widehat{\partial}_I]f - [\mathfrak{D}^r, \widehat{\partial}_{II}]g||_{L^2(\Omega^{d_0})} \\
&\quad \leq C_r\Big(\textstyle\sum_{\ell \leq r-1} ||\mathfrak{D}^\ell \widehat{\partial}_I f - \mathfrak{D}^\ell \widehat{\partial}_{II} g||_{L^2(\Omega^{d_0})} + ||\mathcal{T}\widetilde{x}_I||_{H^r(\Omega)}\{||\widehat{\partial}_I f - \widehat{\partial}_{II} g||_{L^\infty(\Omega^{d_0})} \\
&\qquad + \textstyle\sum_{\ell \leq 2} ||\mathfrak{D}^\ell \widehat{\partial}_I f - \mathfrak{D}^\ell \widehat{\partial}_{II} g||_{L^6(\Omega^{d_0})}\} + \{||\mathcal{T}\widetilde{x}_I - \mathcal{T}\widetilde{x}_{II}||_{H^r(\Omega)} \\
&\qquad + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell\big((S_\varepsilon V_I - S_\varepsilon V_{II}\big)||_{H^{r-\ell}(\Omega)}\}(||\widehat{\partial}_{II} g||_{L^\infty(\Omega^{d_0})} + \sum_{\ell \leq 2} ||\mathfrak{D}^\ell \widehat{\partial}_{II} g||_{L^6(\Omega^{d_0})}) \\
&\qquad + \{||\widetilde{x}_I - \widetilde{x}_{II}||_{H^r(\Omega)} + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell\big(S_\varepsilon V_I - S_\varepsilon V_{II}\big)||_{H^{r-\ell}(\Omega)}\}\sum_{\ell \leq r-1} ||\mathfrak{D}^\ell \widehat{\partial}_{II} g||_{L^2(\Omega^{d_0})}\Big).
\end{aligned} \tag{2.D.10}$$

Proof. We start by writing:

$$\begin{aligned}
&[\mathfrak{D}^r, \widehat{\partial}_I]f - [\mathfrak{D}^r, \widehat{\partial}_{II}]g = \underbrace{-\big((\widehat{\partial}_I \mathfrak{D}^r \widehat{x}_I)\widehat{\partial}_I f - (\widehat{\partial}_{II} \mathfrak{D}^r \widehat{x}_{II})\widehat{\partial}_{II} g\big)}_{I} \\
&+ \textstyle\sum_{\ell_1 + \cdots + \ell_s = r, \ell_i \leq r-1} \underbrace{(\partial \mathfrak{D}^{\ell_1} \widehat{x}_I) \cdots (\partial \mathfrak{D}^{\ell_{s-1}} \widehat{x}_I)(\mathfrak{D}^{\ell_s} \widehat{\partial}_I f) - (\partial \mathfrak{D}^{\ell_1} \widehat{x}_{II}) \cdots (\partial \mathfrak{D}^{\ell_{s-1}} \widehat{x}_{II})(\mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g)}_{II}.
\end{aligned} \tag{2.D.11}$$

We have:

$$||I||_{L^2(\Omega^{d_0})} \leq ||(\widehat{\partial}_I \mathfrak{D}^r \widehat{x}_I - \widehat{\partial}_{II} \mathfrak{D}^r \widehat{x}_{II}) \widehat{\partial}_{II} g||_{L^2(\Omega^{d_0})} + ||(\widehat{\partial}_I \mathfrak{D}^r \widehat{x}_I)(\widehat{\partial}_I f - \widehat{\partial}_{II} g)||_{L^2(\Omega^{d_0})}$$

$$\leq ||\widehat{\partial}_I \mathfrak{D}^r \widehat{x}_I - \widehat{\partial}_{II} \mathfrak{D}^r \widehat{x}_{II}||_{L^2(\Omega^{d_0})} ||\widehat{\partial}_{II} g||_{L^\infty(\Omega^{d_0})} + ||\widehat{\partial}_I \mathfrak{D}^r \widehat{x}_I||_{L^2(\Omega^{d_0})} ||\widehat{\partial}_I f - \widehat{\partial}_{II} g||_{L^\infty(\Omega^{d_0})}$$

$$\leq \Big(||\mathcal{T}(\widehat{x}_I - \widehat{x}_{II})||_{H^r(\Omega^{d_0})} + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell (\widehat{V}_I - \widehat{V}_{II})||_{H^{r-\ell}(\Omega^{d_0})} \Big) ||\widehat{\partial}_{II} g||_{L^\infty(\Omega^{d_0})}$$

$$+ \Big(||\mathcal{T} \widehat{x}_I||_{H^r(\Omega^{d_0})} + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell \widehat{V}_I||_{H^{r-\ell}(\Omega^{d_0})} \Big) ||\widehat{\partial}_I f - \widehat{\partial}_{II} g||_{L^\infty(\Omega^{d_0})}. \quad (2.D.12)$$

In addition, to control II in L^2 one only needs to consider

$$II_1 = (\partial \mathfrak{D}^{\ell_1} \widehat{x}_I) \cdots (\partial \mathfrak{D}^{\ell_{s-1}} \widehat{x}_I)(\mathfrak{D}^{\ell_s} \widehat{\partial}_I f - \mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g), \quad (2.D.13)$$

$$II_2 = (\partial \mathfrak{D}^{\ell_1} \widehat{x}_I - \partial \mathfrak{D}^{\ell_1} \widehat{x}_{II})(\partial \mathfrak{D}^{\ell_2} \widehat{x}_{II}) \cdots (\partial \mathfrak{D}^{\ell_{s-1}} \widehat{x}_{II}) \mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g. \quad (2.D.14)$$

When $r-1 \geq \ell_s \geq 3$ then $\ell_j \leq r-3$ for $j \leq s-1$ and we control the terms involving $\widehat{x}$ in L^∞. Hence

$$||II_1||_{L^2(\Omega^{d_0})} \leq C_r' ||\mathfrak{D}^{\ell_s} \widehat{\partial}_I f - \mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^2(\Omega^{d_0})}, \quad (2.D.15)$$

where

$$C_r' = C_r'\big(M', ||\widehat{x}_I||_{H^r(\Omega^{d_0})}, ||\widehat{x}_{II}||_{H^r(\Omega^{d_0})}, \textstyle\sum_{\ell \leq r-1} ||D_t^\ell \widehat{V}_I||_{H^{r-\ell}(\Omega^{d_0})}, \textstyle\sum_{\ell \leq r-1} ||D_t^\ell \widehat{V}_{II}||_{H^{r-\ell}(\Omega^{d_0})}\big),$$

and

$$||II_2||_{L^2(\Omega^{d_0})} \leq C_r' ||\partial \mathfrak{D}^{\ell_1} \widehat{x}_I - \partial \mathfrak{D}^{\ell_1} \widehat{x}_{II}||_{L^\infty(\Omega^{d_0})} ||\mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^2(\Omega^{d_0})}$$

$$\leq C_r' \Big(||\widehat{x}_I - \widehat{x}_{II}||_{H^r(\Omega^{d_0})} + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell (\widehat{V}_I - \widehat{V}_{II})||_{H^{r-\ell}(\Omega^{d_0})} \Big) ||\mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^2(\Omega^{d_0})}. \quad (2.D.16)$$

Second, when $\ell_j \geq r-2$ for $j = 1, \cdots, s-1$, since $r \geq 5$, there is at most one ℓ_j, say ℓ_1, can be greater than or equal to $r-2$. In this case, we have:

$$||II_1||_{L^2(\Omega^{d_0})} \leq C_r' ||\partial \mathfrak{D}^{\ell_1} \widehat{x}_I||_{L^3(\Omega^{d_0})} ||\mathfrak{D}^{\ell_s} \widehat{\partial}_I f - \mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^6(\Omega^{d_0})}$$

$$\leq C_r' \Big(||\mathcal{T} \widehat{x}_I||_{L^2(\Omega^{d_0})} + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell \widehat{V}_I||_{H^{r-\ell}(\Omega^{d_0})} \Big) ||\mathfrak{D}^{\ell_s} \widehat{\partial}_I f - \mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^6(\Omega^{d_0})}, \quad (2.D.17)$$

where $\ell_s \leq 2$, and

$$||II_2||_{L^2(\Omega^{d_0})} \leq C_r' ||\partial \mathfrak{D}^{\ell_1} \widehat{x}_I - \partial \mathfrak{D}^{\ell_1} \widehat{x}_{II}||_{L^3(\Omega^{d_0})} ||\mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^6(\Omega^{d_0})}$$

$$\leq C_r' \Big(||\mathcal{T}(\widehat{x}_I - \widehat{x}_{II})||_{H^r(\Omega^{d_0})} + \textstyle\sum_{\ell \leq r-1} ||D_t^\ell (\widehat{V}_I - \widehat{V}_{II})||_{H^{r-\ell}(\Omega^{d_0})} \Big) ||\mathfrak{D}^{\ell_s} \widehat{\partial}_{II} g||_{L^6(\Omega^{d_0})}. \quad (2.D.18)$$

This concludes the proof after adapting the Sobolev extension theorem. □

As a consequence, if we take $g = 0$, $\widetilde{x}_I = \widetilde{x}_{II} \equiv \widetilde{x}$, we have:

Corollary 2.D.1. *If $r \geq 5$, there is a constant $C_r = C_r(M_0, ||\widetilde{x}||_{H^r(\Omega)}, \sum_{\ell \leq r-1} ||D_t^\ell V||_{H^{r-\ell}(\Omega)})$ such that:*

$$||[\mathfrak{D}^r, \widehat{\partial}] f||_{L^2(\Omega^{d_0})} \leq C_r \Big(||\mathcal{T} \widetilde{x}||_{H^r(\Omega)} \big(||\widehat{\partial} f||_{L^\infty(\Omega^{d_0})} + \textstyle\sum_{\ell \leq 2} ||\mathfrak{D}^\ell \widehat{\partial} f||_{L^6(\Omega^{d_0})} \big)$$

$$+ \textstyle\sum_{\ell \leq r-1} ||\mathfrak{D}^\ell \widehat{\partial} f||_{L^2(\Omega^{d_0})} \Big). \quad (2.D.19)$$

Here $\mathfrak{D}^r$ be the mixed space-time tangential derivatives defined in Section 2.3.3. In particular, one has:

$$||[\mathfrak{D}^{r-1} D_t, \widehat{\partial}] f||_{L^2(\widehat{\mathcal{D}}_t)} \leq C_r \Big(||\widehat{\partial} f||_{L^\infty(\widehat{\mathcal{D}}_t)} + \textstyle\sum_{\ell \leq 2} ||\mathfrak{D}^\ell \widehat{\partial} f||_{L^6(\widehat{\mathcal{D}}_t)} + \textstyle\sum_{\ell \leq r-1} ||\mathfrak{D}^\ell \widehat{\partial} f||_{L^2(\widehat{\mathcal{D}}_t)} \Big). \quad (2.D.20)$$

The following lemma is similar to the previous one but is better adapted to proving estimates for the wave equation. As in the previous lemma, the point is that the commutator between r derivatives and $\widetilde{\partial}$ is a differential operator of order r with coefficients depending on $r+1$ derivatives of $\widetilde{x}$.

In the following lemma, we will assume that we have the following a priori bound for $\widetilde{x}_I, \widetilde{x}_{II}$:

$$\sum_{|I|+k\leq 3} |D_t^k \partial_y^I \widetilde{x}_I| + |D_t^k \partial_y^I \widetilde{x}_{II}| \leq M. \tag{2.D.21}$$

If we are considering vector fields which do not involve time derivatives, we can instead assume that only:

$$\sum_{|I|\leq 3} |\partial_y^I \widetilde{x}_I| + |D_t^k \partial_y^I \widetilde{x}_{II}| \leq M_0. \tag{2.D.22}$$

Lemma 2.D.4. *Fix $s \geq 0$ and suppose that (2.D.21) holds. If $\mathcal{U} = \mathcal{D}, \mathfrak{D}$ or $\mathcal{U} = \mathcal{T}$, there is a constant $C_s = C_s(M, ||\mathcal{U}^{s-2}\widetilde{x}_I||_{H^2(\Omega)}, ||\mathcal{U}^{s-2}\widetilde{x}_{II}||_{H^2(\Omega)})$ so that if $T^J \in \mathcal{U}^s$, with notation as in Section 2.3.3, then:*

$$||[T^J, \widetilde{\partial}_I]f - [T^J, \widetilde{\partial}_{II}]g||_{L^2} \leq C_s\big(||T^J \widetilde{x}_{II}||_{H^1} + 1\big) \sum_{j\leq s} ||\mathcal{U}^{j-2}(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)||_{H^1}$$

$$+ C_s\big(|||T^J(\widetilde{x}_I - \widetilde{x}_{II})||_{H^1} + ||\mathcal{U}^{s-1}(\widetilde{x}_I - \widetilde{x}_{II})||_{H^1} + ||\mathcal{U}^2(\widetilde{x}_I - \widetilde{x}_{II})||_{C^1_{y,t}}\big) \sum_{j\leq s} ||\mathcal{U}^{j-2}\widetilde{\partial}_{II} g||_{H^1}, \tag{2.D.23}$$

with $H^s = H^s(\Omega)$ and $||\alpha||_{C^1_{y,t}} = \sum_{k+|J|\leq 1} ||\partial_y^J D_t^k \alpha||_{L^\infty([0,T]\times\Omega)}$. If $\mathcal{U} = \{\partial_{y^1}, \partial_{y^2}, \partial_{y^3}\}$, the above estimate holds assuming (2.D.22) holds with M replaced by M_0.

Before proving this lemma we record a few useful instances of it which will be used at several points. Taking $g = 0$ and writing $\widetilde{x} = \widetilde{x}_I$, with $C_s' = C_s'(M, ||\mathcal{T}^{s-2}\widetilde{x}||_{H^2(\Omega)})$, we have:

$$||[\partial_y^I, \widetilde{\partial}]f||_{L^2(\Omega)} \leq C_s(M_0, ||\widetilde{x}||_{H^{s+1}})||\widetilde{\partial} f||_{H^{s-1}(\Omega)}, \qquad |I| = s, \tag{2.D.24}$$

$$||[T^J, \widetilde{\partial}]f||_{L^2(\Omega)} \leq C_s'(||T^J \widetilde{x}||_{H^1(\Omega)} + 1) \sum_{j\leq s-1} ||\mathcal{T}^j \widetilde{\partial} f||_{H^1(\Omega)}, \quad T^J \in \mathcal{T}^s. \tag{2.D.25}$$

Proof. Using (2.D.2), we have:

$$[T^J, \widetilde{\partial}_I]f - [T^J, \widetilde{\partial}_{II}]g = \underbrace{-\big((\widetilde{\partial}_I T^J \widetilde{x}_I)\widetilde{\partial}_I f - (T^J \widetilde{x}_{II})\widetilde{\partial}_{II} g\big)}_{I}$$

$$+ \sum_{J_1+\cdots+J_m=J, |J_i|\le s-1} \underbrace{(\partial T^{J_1}\widetilde{x}_I)\cdots(\partial T^{J_{m-1}}\widetilde{x}_I)T^{J_m}\widetilde{\partial}_I f - (\partial T^{J_1}\widetilde{x}_{II})\cdots(\partial T^{J_{m-1}}\widetilde{x}_{II})T^{J_m}\widetilde{\partial}_{II} g}_{II}, \tag{2.D.26}$$

We control:

$$||(\widetilde{\partial}_I T^J \widetilde{x}_I)\widetilde{\partial}_I f - (\widetilde{\partial}_{II} T^J \widetilde{x}_{II})\widetilde{\partial}_{II} g||_{L^2(\Omega)} \le ||(\widetilde{\partial}_I T^J \widetilde{x}_I - \widetilde{\partial}_{II} T^J \widetilde{x}_{II})\widetilde{\partial}_{II} g||_{L^2(\Omega)}$$

$$+ ||(\widetilde{\partial}_{II} T^J \widetilde{x}_{II})(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g)||_{L^2(\Omega)}. \tag{2.D.27}$$

If $|J| \le 2$, then we control the factors involving $\widetilde{x}_I, \widetilde{x}_{II}$ in L^∞ and the result is bounded by the right-hand side of (2.D.23). If instead $|J| \ge 3$, we control the factors involving f, g in L^∞ and note that since we must have $s \ge 3$, by the Sobolev estimate (2.A.43), we have:

$$||\widetilde{\partial}_{II} g||_{L^\infty(\Omega)} \le C\sum_{j\le 2} ||\mathcal{U}^j \widetilde{\partial}_{II} g||_{H^1(\Omega)} \le C\sum_{j\le s} ||\mathcal{U}^j \widetilde{\partial}_{II} g||_{H^1(\Omega)}, \tag{2.D.28}$$

which is bounded by the right-hand side of (2.D.23). Bounding $||\widetilde{\partial}_I f - \widetilde{\partial}_{II} g||_{L^\infty(\Omega)}$ in the same way shows that the left-hand side of (2.D.27) is controlled by the right-hand side of (2.D.23).

To control II, it suffices to consider

$$II_1 = (\partial T^{J_1}\widetilde{x}_I - \partial T^{J_1}\widetilde{x}_{II})(\partial T^{J_2}\widetilde{x}_{II})\cdots(\partial T^{J_{m-1}}\widetilde{x}_{II})T^{J_m}\widetilde{\partial}_{II} g, \tag{2.D.29}$$

$$II_2 = (\partial T^{J_1}\widetilde{x}_I)\cdots(\partial T^{J_{m-1}}\widetilde{x}_I)T^{J_m}(\widetilde{\partial}_I f - \widetilde{\partial}_{II} g), \tag{2.D.30}$$

where $J_1 + ... + J_m = J$ and $|J_1|, ..., |J_m| \le s-1$. We will just bound $||II_2||_{L^2(\Omega)}$, since the estimate for $||II_1||_{L^2(\Omega)}$ is similar. We start by noting that for each J_i with $|J_i| \le 2, i \le m-1$, we control the corresponding factors of $\widetilde{x}_I$ in L^∞ by the right-hand side of (2.D.23). Rearranging indices, it therefore suffices to control:

$$||(\widetilde{\partial}_I T^{J_1}\widetilde{x}_I)\cdots(\widetilde{\partial}_I T^{J_\ell}\widetilde{x}_I)(T^{J_m}\widetilde{\partial}_I(f-g))||_{L^2(\Omega)}, \tag{2.D.31}$$

$$|J_1|,...,|J_\ell| \geq 3, |J_1|+\cdots+|J_\ell|+|J_m| \leq s-1. \tag{2.D.32}$$

If there are no factors of $\widetilde{x}_I$ present then the result is bounded by $||\widetilde{\partial}T^{J_m}(f-g)||_{L^2(\Omega)}$ and since $J_m \leq s-1$ we control this by the right-hand side of (2.D.23). If there is at least one factor of $\widetilde{x}_I$ present, Note that the conditions on the $|J_k|$ force $|J^m| \leq s-4$ and so we control the last factor in L^∞ by $\sum_{j\leq s-2}||\mathcal{V}^j\widetilde{\partial}(f-g)||_{H^1(\Omega)}$ by the Sobolev estimate (2.A.43). We now use Holder's inequality and the Sobolev embedding (2.A.36) to control:

$$\begin{aligned}||\widetilde{\partial}_I T^{J_1}\widetilde{x}_I\cdots\widetilde{\partial}_I T^{J_\ell}\widetilde{x}_I||_{L^2(\Omega)} &\leq C||\widetilde{\partial}_I T^{J_1}\widetilde{x}_I||_{L^{2\ell}(\Omega)}\cdots||\widetilde{\partial}_I T^{J_\ell}\widetilde{x}_I||_{L^{2\ell}(\Omega)}\\ &\leq C||\widetilde{\partial}_I T^{J_1}\widetilde{x}_I||_{H^1(\Omega)}\cdots||\widetilde{\partial}_I T^{J_\ell}\widetilde{x}_I||_{H^1(\Omega)}. \end{aligned}\tag{2.D.33}$$

We now note that since $|J_1|+...+|J_\ell| \leq s-1$ and $|J_k| \geq 3$ for each $k = 1,...,\ell$, we in fact have $|J_k| \leq s-2$ for each k, and so each of these factors is controlled by the right-hand side of (2.D.23). $\square$

We also need a version with pure time derivatives in the proof of the estimates for the wave equation.

Lemma 2.D.5. *Fix $s \geq 0$. If (2.5.2) holds, there is a constant $C_s = C_s(M, ||\widetilde{x}||_s, ||V||_{\mathcal{X}^s})$ so that*

$$||[D_t^{s+1}, \widetilde{\partial}]f||_{L^2(\Omega)} \leq C_s(||\widetilde{\partial}f||_{s,0} + (||V||_{\mathcal{X}^{s+1}}+1)||\widetilde{\partial}f||_{s-1}). \tag{2.D.34}$$

Proof. By (2.D.26) with $g = 0$ and $T^J = D_t^{s+1}$, we have:

$$[D_t^{s+1}, \widetilde{\partial}_i]f = -(\widetilde{\partial} D_t^{s+1}\widetilde{x})(\widetilde{\partial} f) + \sum\nolimits_{s_1+...+s_m=s+1,\ s_m\geq 1}(\partial D_t^{s_1}\widetilde{x})\cdots(\partial D_t^{s_{m-1}}\widetilde{x})(D_t^{s_m}\widetilde{\partial} f). \tag{2.D.35}$$

We now argue as in the previous lemma, but we want to point out explicitly how the norms of V arise. We write the first term as $-(\widetilde{\partial} D_t^s S_\varepsilon V)(\widetilde{\partial} f)$. If $s \leq 2$ then we control the first factor in L^∞ since $S_\varepsilon : L^\infty \to L^\infty$. If instead $s \geq 3$, we control the second factor using Sobolev embedding, $||\widetilde{\partial} f||_{L^\infty(\Omega)} \leq C||\widetilde{\partial} f||_{H^2(\Omega)} \leq C||\widetilde{\partial} f||_{s-1}$, and now we note that $||\widetilde{\partial} D_t^s S_\varepsilon V||_{L^2(\Omega)} \leq C(M)||V||_{\mathcal{X}^{s+1}}$, using that $S_\varepsilon : L^2 \to L^2$.

The terms in the sum can be controlled using essentially the same argument as in the previous lemma. Rearranging indices it suffices to control:

$$||(\widetilde{\partial} D_t^{s_1}\widetilde{x})\cdots(\widetilde{\partial} D_t^{s_j}\widetilde{x})(D_t^{s_m}\widetilde{\partial} f)||_{L^2(\Omega)}, \tag{2.D.36}$$

$$s_\ell \geq 3, \ell = 1, ..., j, s_1 + \cdots s_j + s_m \leq s, s_m \geq 1. \tag{2.D.37}$$

If there are no factors of $\widetilde{x}$ present then we control this by $||D_t^{s_m}\widetilde{\partial} f||_{L^2(\Omega)} \leq ||\widetilde{\partial} f||_{s,0}$. and if there is at least one factor of $\widetilde{x}$ present then we must have $s_m \leq s-3$ and so we can control $||D_t^{s_m}\widetilde{\partial} f||_{L^\infty(\Omega)} \leq C||D_t^{s_m}\widetilde{\partial} f||_{H^2(\Omega)} \leq C||\widetilde{\partial} f||_{s-1}$. When $j = 1$ the result is obvious since $||\widetilde{\partial} D_t^{s_1}\widetilde{x}||_{L^2(\Omega)} \leq C||\widetilde{x}||_{s_1+1}$ and $s_1 \leq s-1$. When $j \geq 2$ we have by Sobolev embedding (2.A.36):

$$||(\widetilde{\partial} D_t^{s_1}\widetilde{x})\cdots(\widetilde{\partial} D_t^{s_j}\widetilde{x})||_{L^2(\Omega)} \leq C||\widetilde{\partial} D_t^{s_1}\widetilde{x}||_{L^{2j}(\Omega)}\cdots||\widetilde{\partial} D_t^{s_j}\widetilde{x}||_{L^{2j}(\Omega)}$$

$$\leq C||\widetilde{\partial} D_t^{s_1}\widetilde{x}||_{H^1(\Omega)}\cdots||\widetilde{\partial} D_t^{s_j}\widetilde{x}||_{H^1(\Omega)}. \tag{2.D.38}$$

Since each s_ℓ must satisfy $s_\ell \leq s-3$, each of these factors is bounded by $C(M)||\widetilde{x}||_{s-1}$, as required. ∎

We also need to use the following commutator estimates in $\widetilde{\mathcal{D}}_t$.

Lemma 2.D.6. *Let $r \geq 7$ and $k+\ell \leq r+1$ with $k \geq 2$, we have:*

$$||[D_t^{k-1}, \widetilde{\partial}]\varphi||_{H^\ell(\mathcal{D}_t)} \leq P\big(\textstyle\sum_{s\leq k-2}||D_t^s S_\varepsilon V||_{H^{r-s}(\widetilde{\mathcal{D}}_t)}\big)\sum_{s\leq k-2}||D_t^s\varphi||_{H^{r-s}(\widetilde{\mathcal{D}}_t)}, \tag{2.D.39}$$

and for $k+\ell = r$, we have:

$$||[D_t^{k-1}, \widetilde{\Delta}]\varphi||_{H^\ell(\mathcal{D}_t)} \leq P\big(\textstyle\sum_{s\leq k-2}||D_t^s S_\varepsilon V||_{H^{r-s}(\widetilde{\mathcal{D}}_t)}\big)\sum_{s\leq k-2}||D_t^s\varphi||_{H^{r-s}(\widetilde{\mathcal{D}}_t)}. \tag{2.D.40}$$

Proof. It is not hard to compute that $[D_t^{k-1}, \widetilde{\partial}]$ consists of terms of the following forms:

$$(\widetilde{\partial}D_t^{s_1}S_\varepsilon V)\cdots(\widetilde{\partial}D_t^{s_{n-1}}S_\varepsilon V)(\widetilde{\partial}D_t^{s_n}\varphi), s_1+\cdots+s_n = k-n,\ \ n\geq 2, \tag{2.D.41}$$

so (2.D.39) follows. On the other hand, (2.D.40) follows after noting that $[D_t^{k-1}, \widetilde{\Delta}]$ consists of terms of the following form:

$$(\widetilde{\partial}D_t^{s_3}S_\varepsilon V)\cdots(\widetilde{\partial}D_t^{s_n}S_\varepsilon V)(\widetilde{\partial}^2 D_t^{s_1}S_\varepsilon V)(\widetilde{\partial}D_t^{s_2}\varphi), s_1+\cdots+s_n = k-n,\ \ n\geq 2, \tag{2.D.42}$$

$$(\widetilde{\partial}D_t^{s_3}S_\varepsilon V)\cdots(\widetilde{\partial}D_t^{s_n}S_\varepsilon V)(\widetilde{\partial}D_t^{s_1}S_\varepsilon V)(\widetilde{\partial}^2 D_t^{s_2}\varphi), s_1+\cdots+s_n = k-n,\ \ n\geq 2. \qquad \square$$

We now prove some estimates which are used in Sections 2.6 and 2.8 to control the terms on the right-hand side of the various wave equations. For these estimates we will assume the following bound for φ:

$$\textstyle\sum_{k+|J|\leq 3}|D_t^k\partial_y^J\widetilde{\partial}\varphi| + |D_t^k\varphi| \leq L. \tag{2.D.43}$$

Lemma 2.D.7. *If the equation of state satisfies (2.1.10) for all $j \geq k+\ell \equiv s$, then there is a constant C depending only on c_1, c_2 and a polynomial P so that:*

$$||D_t^k(e'(\varphi)D_t^2\varphi) - e'(\varphi)(D_t^{k+2}\varphi)||_{H^\ell} \leq CL||D_t^{k+1}\varphi||_{H^\ell} + P(L, ||\varphi||_s). \tag{2.D.44}$$

Proof. We just prove the $\ell = 0$ case since $\ell \geq 1$ is similar. The main term in $D_t^k(e'(\varphi)D_t^2\varphi) -$

$e'(\varphi)(D_t^{k+2}\varphi)$ is:

$$e''(\varphi)(D_t\varphi)D_t^{k+1}\varphi, \tag{2.D.45}$$

and the remaining terms are of the form

$$e^{(m)}(\varphi)D_t^{k_1}\varphi\cdots D_t^{k_m}\varphi, \quad 2\leq m\leq k, \quad k_1+\cdots+k_m=k+1, k_j\leq k, 1\leq j\leq m. \tag{2.D.46}$$

The term (2.D.45) is bounded by the right-hand side of (2.D.44). To bound (2.D.46), we note that if $k_j \leq 3$ for $1\leq j\leq m$ all of the terms are bounded by L. If there are any terms with $k_j\leq k-2$ then by Sobolev embedding we have $||D_t^{k_j}\varphi||_{L^\infty}\leq C||\varphi||_k$. Therefore it just remains to consider the case that there is at least one j with $k_j\geq \max(4, k-1)$ and in fact there can be at most one such term since we also have $k_j\leq k$ for each j. In this case we put the corresponding factor in L^2 and this proves (2.D.44). $\square$

The following estimate is nearly the same as (2.D.44) but will be used in Section 2.F.2 to bound quantities of the form $e'(f)g$ when we know that f is smoother than g.

Lemma 2.D.8. *Under the hypotheses of Lemma 2.D.7, if $k+\ell = s$ then there is a constant C depending only on c_1, c_2 on is a polynomial P so that if φ satisfies (2.D.43):*

$$||D_t^k e'(\varphi)||_{H^\ell}\leq C\big(||D_t^k\varphi||_{H^\ell}+P(L,||\varphi||_{s-1})\big). \tag{2.D.47}$$

Proof. The proof is similar to the proof of Lemma 2.D.7. If $|I|=\ell$ then $\partial_y^I D_t^k e'(\varphi)$ is a sum of terms

$$e^m(\varphi)(\partial_y^{I_1}D_t^{k_1}\varphi)\cdots(\partial_y^{I_m}D_t^{k_m}\varphi)\varphi), \qquad |I_1|+...+|I_m|=|I|, \quad k_1+...+k_m=k. \tag{2.D.48}$$

Using Sobolev embedding as in the proof of Lemma 2.D.7 gives the result. $\square$

We will also need estimates for the derivatives of $\mathcal{F} = \mathcal{F}^1 + \mathcal{F}^2$, where

$$\mathcal{F}^1 = -(\widetilde{\partial}_i S_\varepsilon V^j)(\widetilde{\partial}_j V^i), \qquad \mathcal{F}^2 = -e''(h)(D_t h)^2 - \rho(h). \tag{2.D.49}$$

Lemma 2.D.9. *If (2.5.2) holds and h satisfies (2.D.43), then for $k \geq 1$:*

$$||D_t^k \mathcal{F}_1||_{L^2} \leq C(M, ||\partial V||_{L^\infty})\big(||D_t^k V||_{H^1} + P(||V||_{\mathcal{X}^k})\big), \tag{2.D.50}$$

$$||D_t^k \mathcal{F}_2||_{L^2} \leq CL||D_t^{k+1} h||_{L^2} + P(L, ||h||_{k,0}). \tag{2.D.51}$$

For $k \geq 0$, writing $k + \ell = s$, we also have:

$$||D_t^k \mathcal{F}_1||_{H^\ell} \leq C(M, ||\partial V||_{L^\infty})\big(||D_t^k V||_{H^{\ell+1}} + ||\widetilde{x}||_{H^{\ell+1}} + P(||V||_s, ||\widetilde{x}||_{H^\ell})\big), \tag{2.D.52}$$

$$||D_t^k \mathcal{F}_2||_{H^\ell} \leq CL||D_t^{k+1} h||_{H^\ell} + P(L, ||h||_s). \tag{2.D.53}$$

Proof. We we just prove the estimate (2.D.52). The estimate (2.D.50) follows in a similar manner and the estimates (2.D.51), (2.D.53) follow as in the previous lemma. The case $k = 0$ can be handled using interpolation and the estimates (2.D.6). When $k \geq 1$, we have:

$$\begin{aligned}\partial^\ell D_t^k \mathcal{F} &= \sum_{l_1 + \cdots + l_n = k,\, n \geq 2} C^k_{l_1 \cdots l_n} \partial^\ell \big((\widetilde{\partial}_{j_1} D_t^{l_1} S_\epsilon V^{j_2}) \cdots (\widetilde{\partial}_{j_{n-1}} D_t^{l_{n-1}} S_\epsilon V^{j_n})(\widetilde{\partial}_{j_n} D_t^{l_n} V^{j_1})\big) \\ &= \sum_{l_1 + \cdots + l_n = k,\ \sum |\beta_j| + |\gamma_j| = \ell - 1} \widetilde{C}^k_{l_1 \cdots l_n} (\partial^{\beta_1} A^{a_1}_{\ j_1}) \cdots (\partial^{\beta_n} A^{a_n}_{\ j_n})(\partial_{a_1} \partial^{\gamma_1} D_t^{l_1} S_\varepsilon V^{j_2}) \cdots (\partial_{a_{j_{n-1}}} \partial^{\gamma_{n-1}} D_t^{l_{n-1}} S_\varepsilon V^{j_n})(\partial_{a_n} \partial^{\gamma_n} D_t^{l_n} V^{j_1}),\end{aligned} \tag{2.D.54}$$

where we have used (2.D.3) repeatedly. The leading order term is of the form

$$A^a_{\ i}(\partial^\alpha D_t^k \partial_a S_\varepsilon V^j)(\widetilde{\partial}_j V^i) + (\partial^\alpha A^a_{\ i})(D_t^k \partial_a S_\varepsilon V^j)(\widetilde{\partial}_j V^j).$$

We bound the first term by:

$$C(M')||\partial V||_{L^\infty}||D_t^k V||_{H^\ell(\Omega)}, \tag{2.D.55}$$

and we bound the second term by:

$$P(||V(t)||_{\mathcal{X}^s}, ||\widetilde{x}(t)||_{H^{r-1}(\Omega)}), \tag{2.D.56}$$

when $k \geq 1$, and

$$C(M')||\partial V||_{L^\infty}||\widetilde{x}||_{H^r(\Omega)}, \tag{2.D.57}$$

when $k = 0$. The lower order terms in (2.D.54) is controlled via Sobolev embedding. $\square$

Writing $\mathcal{F}_J = -(\widetilde{\partial}_i S_\varepsilon V_J^\ell)(\widetilde{\partial}_\ell V_J^i)$ for $J = I, II$, a simple modification of the proof of Lemma 2.D.9 gives:

Lemma 2.D.10. *Suppose that* (2.5.2) *holds and let $s = k + \ell$. Then there is a continuous, positive function $C_s = C_s(M, ||V_I||_s, ||V_{II}||_s, ||\widetilde{x}_I||_{H^{s+1}}, ||\widetilde{x}_{II}||_{H^{s+1}})$ so that:*

$$\begin{aligned}||D_t^k(\mathcal{F}_I^1 - \mathcal{F}_{II}^1)||_{H^\ell} \leq C_s\big(&||D_t^k V_I - D_t^k V_{II}||_{H^{\ell+1}} \\ &+ ||\widetilde{x}_I - \widetilde{x}_{II}||_{H^{\ell+1}} + ||V_I - V_{II}||_s + ||\widetilde{x}_I - \widetilde{x}_{II}||_{C^4_{x,t}(\Omega)}\big),\end{aligned} \tag{2.D.58}$$

$$||F^2(h_I) - F^2(h_{II})||_{s,0} \leq C_s\big(||h_I - h_{II}||_{s+1,0} + ||h_I - h_{II}||_s + ||h_I - h_{II}||_{C^3_{x,t}}\big), \tag{2.D.59}$$

$$||F^2(h_I) - F^2(h_{II})||_{s-1} \leq C_s\big(||h_I - h_{II}||_s + ||h_I - h_{II}||_{C^3_{x,t}}\big). \tag{2.D.60}$$

2.E Existence of a sequence of compatible data for the smoothed problem

In this section, our goal is to prove:

Theorem 2.E.1. *Suppose that $V_0, h_0 \in H^r$, $x_0 \in H^{r+1}$ satisfy the compatibility conditions for Euler's equations* (2.2.25) *to order $r-1 \geq 7$. Then there is a sequence of data $V_0^\varepsilon, h_0^\varepsilon \in H^r$, $x_0^\varepsilon \in H^{r+1}$ satisfying the compatibility conditions for the smoothed Euler's equations* (2.4.15) *to order $r-1$, and*

$(V_0^\varepsilon, h_0^\varepsilon, x_0^\varepsilon) \to (V_0, h_0, x_0)$ *as* $\varepsilon \to 0$.

In the next section, we prove that if the compatibility conditions to order $r-1$ hold, given sufficiently regular V, the wave equation (2.2.10) has a solution $h(t,\cdot) \in H^r(\Omega)$ with $D_t h(t,\cdot) \in H^{r-1}(\Omega), ..., D_t^{r-1} h(t,\cdot) \in H_0^1(\Omega), D_t^r h(t,\cdot) \in L^2(\Omega)$ for $t > 0$. We modify the approach of Lindblad-Luo [10] to construct functions $u_{-1}^\varepsilon, u_0^\varepsilon$ so that with $V_0^\varepsilon = V_0 + \partial_{x_0} u_{-1}^\varepsilon$, $h_0^\varepsilon = h_0 + u_0^\varepsilon$, and $x_0^\varepsilon = x_0$, the initial data $V_0^\varepsilon, h_0^\varepsilon, x_0^\varepsilon$ satisfy the compatibility conditions (2.4.15). It will be convenient to reformulate the conditions used in Sections 2.2.3 and 2.4.3 in a slightly more explicit way. Suppose that $\hat{x} = \sum x_k t^k/k!$, $\hat{V} = \sum V_k t^k/k!$, $\hat{h} = \sum V_k t^k/k!$ are formal power series solutions at $t = 0$ to (2.1.2)-(2.1.1) with $\hat{x}|_{t=0} = x_0$ and $D_t^{\ell+1}\hat{x}|_{t=0} = D_t^\ell \hat{V}|_{t=0}$ and $\hat{x}_\varepsilon = \sum x_k^\varepsilon t^k/k!$, $\hat{V}_\varepsilon = \sum V_k^\varepsilon t^k/k!$, $\hat{h}_\varepsilon = \sum h_k^\varepsilon t^k/k!$ are power series solutions at $t = 0$ to the smoothed problem (2.4.11)-(2.4.12) with $\hat{x}|_{t=0} = x_0$ and $D_t^{\ell+1}\hat{x}^\varepsilon|_{t=0} = D_t^\ell \hat{V}^\varepsilon|_{t=0}$. Define:

$$F_k = \big([D_t^k, \hat{\Delta}]\hat{h} + D_t^k(\hat{\partial}_i \hat{V}^j)(\hat{\partial}_j \hat{V}^i)\big)\big|_{t=0}, \tag{2.E.1}$$

$$G_k = \big(D_t^{k+1}(e'(\hat{h})D_t\hat{h}) - e'(\hat{h})D_t^{k+2}\hat{h} + D_t^k \rho[\hat{h}]\big)\big|_{t=0}, \tag{2.E.2}$$

$$F_k^\varepsilon = \big([D_t^k, \hat{\hat{\Delta}}]\hat{h}_\varepsilon + D_t^k(\hat{\hat{\partial}}_i S_\varepsilon \hat{V}_\varepsilon^j)(\hat{\hat{\partial}}_j \hat{V}_\varepsilon^i)\big)\big|_{t=0}, \tag{2.E.3}$$

$$G_k^\varepsilon = \big(D_t^{k+1}(e'(\hat{h}_\varepsilon)D_t\hat{h}_\varepsilon) - e'(\hat{h}_\varepsilon)D_t^{k+2}\hat{h}_\varepsilon + D_t^k \rho[\hat{h}_\varepsilon]\big)\big|_{t=0}, \tag{2.E.4}$$

as well as:

$$C_k = [D_t^k, \hat{\partial}](\hat{h} + \phi[\hat{x}, \hat{h}])|_{t=0}, \qquad C_k^\varepsilon = [D_t^k, \hat{\hat{\partial}}](\hat{h}_\varepsilon + \phi[\hat{x}_\varepsilon, \hat{h}_\varepsilon]|_{t=0}. \tag{2.E.5}$$

Here, we are writing:

$$\hat{\partial}_i = \frac{\partial y^a}{\partial \hat{x}^i}\frac{\partial}{\partial y^a}, \qquad \hat{\hat{\partial}}_i = \frac{\partial y^a}{\partial \hat{\hat{x}}^i}\frac{\partial}{\partial y^a}, \qquad \hat{\Delta} = \textstyle\sum_{i=1}^3 \hat{\partial}_i^2, \qquad \hat{\hat{\Delta}} = \textstyle\sum_{i=1}^3 \hat{\hat{\partial}}_i^2. \tag{2.E.6}$$

We are also writing $\phi[x,h]$ for the map $x, h \mapsto \phi$ defined in (2.2.6).

Taking the divergence of Euler's equation (2.1.1) at $t = 0$ and subtracting it from the continuity equation (2.1.2) at $t = 0$ and performing the same manipulations to (2.4.11) and (2.4.12) gives that the coefficients $x_k, V_k, h_k, x_k^\varepsilon, V_k^\varepsilon, h_k^\varepsilon$ must satisfy the relations:

$$x_k = V_{k-1}, \qquad V_k = -\partial_{x_0} H_{k-1} + C_k, \qquad e'(h_0) h_{k+2} = \Delta h_k + F_k + G_k, \tag{2.E.7}$$

$$x_k^\varepsilon = V_{k-1}^\varepsilon, \qquad V_k^\varepsilon = -\partial_{x_0} H_{k-1}^\varepsilon + C_k^\varepsilon, \qquad e'(h_0^\varepsilon) h_{k+2}^\varepsilon = \Delta h_k^\varepsilon + F_k^\varepsilon + G_k^\varepsilon, \tag{2.E.8}$$

for $k \geq 1$, with $H_{k-1} = h_{k-1} + \phi_{k-1}, H_{k-1}^\varepsilon = h_{k-1}^\varepsilon + \phi_{k-1}^\varepsilon$, and where $\phi_\ell = D_t^\ell \phi[\hat{x}, \hat{h}]|_{t=0}, \phi_\ell^\varepsilon = D_t^\ell \phi[\hat{x}_\varepsilon, \hat{h}_\varepsilon]$.

Expanding out the various definitions and replacing x_k with V_{k-1} for $k \geq 1$, it follows that:

$$F_k = F_k[x_0, V_0, ..., V_k, h_0, ..., h_{k-1}], \qquad G_k = G_k[h_0, ..., h_{k+1}], \tag{2.E.9}$$

$$\phi_k = K_k[x_0, V_0, ..., V_k, h_0, ..., h_k], \qquad C_k = C_k[x_0, V_0, ..., V_{k-1}, H_0, ..., H_{k-1}]. \tag{2.E.10}$$

for functionals K_k and where these functionals depend on space derivatives of their arguments, and similarly:

$$F_k^\varepsilon = F_k^\varepsilon[x_0, V_0^\varepsilon, ..., V_k^\varepsilon, h_0^\varepsilon, ..., h_{k-1}^\varepsilon], \qquad G_k^\varepsilon = G_k^\varepsilon[h_0^\varepsilon, ..., h_{k+1}^\varepsilon], \tag{2.E.11}$$

$$\phi_k^\varepsilon = K_k^\varepsilon[x_0, V_0^\varepsilon, ..., V_k^\varepsilon, h_0^\varepsilon, ..., h_k^\varepsilon], \qquad C_k^\varepsilon = C_k^\varepsilon[x_0, V_0^\varepsilon, ..., V_{k-1}^\varepsilon, H_0^\varepsilon, ..., H_{k-1}^\varepsilon]. \tag{2.E.12}$$

The formulas (2.E.10) combined with the second identity in (2.E.7) shows that V_k can be expressed entirely in terms of x_0, V_0 and $h_0, ..., h_{k-1}$ and similarly V_k^ε can be expressed entirely in terms of $x_0, V_0, h_0^\varepsilon, ..., h_{k-1}^\varepsilon$. Consequently we will eliminate V_k, V_k^ε for $k \geq 1$ from our equations and abuse notation slightly and write:

$$F_k = F_k[x_0, V_0, h_0, ..., h_{k-1}], \qquad F_k^\varepsilon = F_k^\varepsilon[x_0, V_0^\varepsilon, h_0^\varepsilon, ..., h_{k-1}^\varepsilon]. \tag{2.E.13}$$

2.E.0.1 The perturbative system

We start by considering the following system, with $\Delta = \sum_{i=1}^{3} \partial_i^2$:

$$\Delta u_{-1}^{\varepsilon} = -e'(h_0 + u_0^{\varepsilon})u_1^{\varepsilon}, \qquad \text{in } \Omega, \tag{2.E.14}$$

$$\Delta u_k^{\varepsilon} = F_k - \widetilde{F}_k^{\varepsilon} + G_k - \widetilde{G}_k^{\varepsilon} + e'(h_0 + u_0^{\varepsilon})u_{k+2}^{\varepsilon}, \qquad \text{in } \Omega, \text{ for } k = 0, ..., r-2, \tag{2.E.15}$$

$$u_k^{\varepsilon} = 0, \qquad \text{on } \partial\Omega, \text{ for } k = -1, ..., r-2, \tag{2.E.16}$$

with $\widetilde{F}_k^{\varepsilon} = \widetilde{F}_k^{\varepsilon}[u_{-1}^{\varepsilon}, ..., u_k^{\varepsilon}], \widetilde{G}_k^{\varepsilon} = \widetilde{G}_k^{\varepsilon}[u_0^{\varepsilon}, ..., u_{k+1}^{\varepsilon}]$ defined by:

$$\widetilde{F}_k^{\varepsilon} = F_k^{\varepsilon}[x_0, V_0 + \widetilde{\partial} u_{-1}^{\varepsilon}, h_0 + u_0^{\varepsilon}, ..., h_{k-1} + u_{k-1}^{\varepsilon}], \quad \widetilde{G}_k^{\varepsilon} = G_k^{\varepsilon}[h_0 + u_0^{\varepsilon}, ..., h_{k+1} + u_{k+1}^{\varepsilon}], \tag{2.E.17}$$

and with the convention that $u_\ell^{\varepsilon} = 0$ for $\ell \geq r-1$.

Suppose for the moment that this system has a solution $(u_{-1}^{\varepsilon}, ..., u_{r-1}^{\varepsilon})$. We claim that with $V_0^{\varepsilon} = V_0 + \partial_{x_0} u_{-1}^{\varepsilon}$ and $h_0^{\varepsilon} = h_0 + u_0^{\varepsilon}$, the initial data $(V_0^{\varepsilon}, h_0^{\varepsilon})$ satisfy the compatibility conditions (2.4.19) for the smoothed problem to order $r-1$. Indeed, because $h_0 = 0$ on $\partial\Omega$ and because of the boundary condition (2.E.16) we have that $h_0^{\varepsilon} = 0$ on $\partial\Omega$. To see that $h_1^{\varepsilon} = 0$ on $\partial\Omega$, we note that by construction:

$$e'(h_0^{\varepsilon})h_1^{\varepsilon} = -\operatorname{div} V_0^{\varepsilon} = -\operatorname{div} V_0 - \Delta u_{-1}^{\varepsilon} = e'(h_0)h_1 + e'(h_0^{\varepsilon})u_1^{\varepsilon}. \tag{2.E.18}$$

By the compatibilty conditions for V_0, h_0, we have $h_1 = 0$ on $\partial\Omega$ and by construction $u_1^{\varepsilon} = 0$ on $\partial\Omega$ and so the first compatibility condition (2.4.15) holds as well. Using the definitions of h_2^{ε}, h_2 from (2.E.7),(2.E.8), we have:

$$e'(h_0^{\varepsilon})h_2^{\varepsilon} = \Delta h_0 + \Delta u_0^{\varepsilon} + F_0^{\varepsilon} + G_0^{\varepsilon} = \Delta h_0 + F_0 + G_0 + e'(h_0^{\varepsilon})u_2^{\varepsilon} = e'(h_0)h_2 + e'(h_0^{\varepsilon})u_2^{\varepsilon}, \tag{2.E.19}$$

By the compatibility conditions, $h_2 = 0$ on $\partial\Omega$ and this combined with the boundary condition

(2.E.16) shows that $h_2^\varepsilon = 0$ on $\partial\Omega$ as well. In general, this construction gives that:

$$e'(h_0^\varepsilon)h_k^\varepsilon = e'(h_0)h_k + e'(h_0^\varepsilon)u_k^\varepsilon,\ k = 0, ..., r-2, \quad e'(h_0^\varepsilon)h_k^\varepsilon = e'(h_0)h_k,\ \ k = r-1, r, \quad (2.E.20)$$

from which it immediately follows that the compatibility condition of order $r-1$ holds for the smoothed problem so long as the compatibility condition of order $r-1$ holds for the original problem.

Because e' is assumed to be small, a simplified model for the above system is the following:

$$\Delta w_{-1} = \kappa w_1, \quad \Delta w_k = \textstyle\sum_{\ell \le k-1} A_k^\ell w_\ell + f_k + \kappa w_{k+2}, \quad k = 0, ..., N, \qquad \text{in } \Omega, \qquad (2.E.21)$$

with the boundary condition $w_k = 0$ on $\partial\Omega$ for all k. Here, A_k^ℓ, f_k are given functions, κ is a small parameter and we are writing $w_\ell = 0$ for $\ell \ge N+1$. When $\kappa = 0$, this system is lower-triangular and can be solved directly by successively solving for $w_0, w_1, ...$. To solve the model system (2.E.21) for nonzero but small κ, one can use the following iteration: $w_k^0 \equiv 0$ for all k and then, given $w_k^{\nu-1}$, solve the following system for w_k^ν:

$$\Delta w_{-1}^\nu = \kappa w_2^{\nu-1}, \quad \Delta w_k^\nu = \textstyle\sum_{\ell \le k-1} A_k^\ell w_\ell^\nu + f_k + \kappa w_{k+2}^{\nu-1}, \qquad k = 0, ..., N-1, \qquad (2.E.22)$$

with $w_k^\nu = 0$ on $\partial\Omega$. Writing $W_k^\nu = w_k^\nu - w_k^{\nu-1}$, by standard elliptic theory there are estimates of the form:

$$||W_k^\nu||_{H^{s-k}} \le C\big(\textstyle\sum_{\ell \le k-1} ||W_\ell^\nu||_{H^{s-k-2}} + \kappa ||W^{\nu-1}||_{H^{s-k-2}}\big), \qquad k = -1...., N, \qquad (2.E.23)$$

where C depends on norms of the coefficients A. Iterating this estimate leads to an inequality of the form:

$$||W_k^\nu||_{s-k} \le \textstyle\sum_{\mu=1}^{\nu-1} (C\kappa)^\mu. \qquad (2.E.24)$$

For κ sufficiently small, the sequence w^ν converges as $\nu \to \infty$ to a solution $w = (w_{-1}, \dots, w_N)$

satisfying (2.E.21).

2.E.0.2 The iteration to solve the system

In order to solve the system (2.E.14)-(2.E.16), we will use the following iteration. We set $u_k^0 \equiv 0$ in Ω for $k = -1, ..., r$ and for $\nu \geq 1$, we define $u_{-1}^\nu, ..., u_r^\nu$ by $u_{r-1}^\nu = u_r^\nu = 0$ and:

$$\Delta u_{-1}^\nu = -e'(h_0 + u_0^{\nu-1})u_1^{\nu-1}, \qquad \text{in } \Omega, \tag{2.E.25}$$

$$\Delta u_k^\nu = F_k - F_k^\nu + G_k - G_k^{\nu-1} + e'(h_0 + u_0^{\nu-1})u_{k+2}^{\nu-1}, \qquad \text{in } \Omega, \text{ for } k = 0, ..., r-2, \tag{2.E.26}$$

$$u_k^\nu = 0, \qquad \text{on } \partial\Omega, \text{ for } k = -1, ..., r-2, \tag{2.E.27}$$

where we are writing:

$$F_k^\nu = \widetilde{F}_k^\varepsilon[u_{-1}^\nu, ..., u_{k-1}^\nu], \qquad G_k^{\nu-1} = \widetilde{G}_k^\varepsilon[u_0^{\nu-1}, ..., u_{k+1}^{\nu-1}]. \tag{2.E.28}$$

Let $u^\nu = (u_{-1}^\nu, ..., u_r^\nu)$. To see that this system has a solution u^ν given $u^{\nu-1}$, one just uses the fact that it is lower-triangular in u^ν; first solve (2.E.25) for u_{-1}^ν and then solve (2.E.26)-(2.E.27) successively for $k = 0, 1, ..., r-2$.

We will prove that the sequence u^ν is uniformly bounded in ν in the norm

$$||u^\nu||_r = ||\partial_{x_0} u_{-1}^\nu||_{H^r(\Omega)} + ||u_{-1}^\nu||_{H^r(\Omega)} + \sum_{k=0}^{r-2} ||u_k^\nu||_{H^{r-k}(\Omega)}. \tag{2.E.29}$$

Set $E_0 = ||V_0||_{H^r}^2 + ||h_0||_{H^r}^2 + ||x_0||_{H^{r+1}}^2$. In the following sections we will prove:

Proposition 2.E.1. *Fix $r \geq 8$. There is a continuous function C_r so that if u^ν satisfies* (2.E.25)-(2.E.27)*, then:*

$$||u^\nu||_r \leq C_r(E_0, ||u^{\nu-1}||_{r-1})(\kappa ||u^{\nu-1}||_r + \varepsilon), \tag{2.E.30}$$

and there is a continuous function D_r so that:

$$||u^\nu - u^{\nu-1}||_r \leq D_r(E_0, ||u^\nu||_r, ||u^{\nu-1}||_r)\kappa||u^{\nu-1} - u^{\nu-2}||_r. \tag{2.E.31}$$

Let us now explain why one should expect estimates of this form. The estimates (2.E.30) follow from elliptic estimates applied to the system (2.E.25)-(2.E.27) and will ultimately follow from estimates for $F_k - F_k^\nu, G_k - G_k^{\nu-1}$ in Sobolev spaces. Let us consider the $k = 0$ case. Using that $\hat{x}|_{t=0} = x_0$ we have:

$$F_0 - F_0^\nu = (\partial_i V_0^j)(\partial_j V_0^i) - (\partial_i S_\varepsilon(V_0^j + \delta^{jk}\partial_k u_{-1}^\nu)(\partial_j(V_0^i + \delta^{ik}\partial_k u_{-1}^\nu)). \tag{2.E.32}$$

Expanding this out generates several terms but let us just consider two of them:

$$\partial_i V_0^j(\delta^{ij}\partial_j\partial_k u_{-1}^\nu) \quad \text{and} \quad (\partial_i V_0^j - \partial_i S_\varepsilon V_0^j)(\partial_j V_0^i). \tag{2.E.33}$$

To control the $L^2(\Omega)$ norm, say, of the first term we use the equation (2.E.25) and standard elliptic theory to control $||u_{-1}^\nu||_{H^2(\Omega)} \leq C||e'(u_0^{\nu-1})u_1^{\nu-1}||_{L^2(\Omega)}$. With $\kappa \geq \sup|e'|$ this type of term can be bounded by the first term in (2.E.30). Also, we have $||V_0 - S_\varepsilon V_0||_{H^1(\Omega)} \leq C\varepsilon||V_0||_{H^2(\Omega)}$ so the second type of term can be bounded by the second term in (2.E.30). Assuming that (2.E.30)-(2.E.31) hold for the moment, we give the proof:

Proof of Theorem 2.E.1. With the function C_r from Proposition 2.E.1, take $C_0 = \max_{z\in[0,1]} C_r(E_0, z)$. Also take κ so small that $2\kappa C_0 \leq 1$ and ε so small that $2\varepsilon\sum_{\mu=0}^\infty(\kappa C_0)^\mu \leq 1$. Since $u^0 = 0$, it follows from (2.E.30) that $||u^1||_r \leq C_0\varepsilon$. By induction it then follows that $||u^\nu||_r \leq \varepsilon\sum_{\mu=0}^\nu(\kappa C_0)^\mu$, and by the assumption on κ the sum on the right-hand side is uniformly bounded as $\nu \to \infty$.

Next, with the function D_r from Proposition 2.E.1, take

$$D_0 = \max_{z_1,z_2\in[0,1]} D_r(E_0, z_1, z_2).$$

By induction and (2.E.31) it follows that $||u^\nu - u^{\nu-1}||_r \le \varepsilon(\kappa D_0)^\nu$. Therefore, u^ν is a Cauchy sequence and so it converges to some limit u which satisfies the perturbative system (2.E.14)-(2.E.16) by construction. To prove the second point in the theorem, taking $\nu \to \infty$ in the estimate for u^ν that we just proved shows $||u||_r \le \varepsilon \sum_{\mu=0}^{\infty}(\kappa C_0)^\mu \le \varepsilon$. □

We will use the following estimate, which is a straightforward consequence of the elliptic estimate (2.5.8) at $t = 0$: If $s \ge 2$, there is a constant $C_s = C_s(||x_0||_{H^{s+1}})$ so that if $f = 0$ on $\partial\Omega$, then:

$$||\partial_{x_0} f||_{H^s} \le C_s ||\Delta_{x_0} f||_{H^{s-1}}, \qquad ||f||_{H^s} \le C_s ||\Delta_{x_0} f||_{H^{s-2}}. \tag{2.E.34}$$

The estimates (2.E.30) and (2.E.31) follow after repeatedly applying the next lemma:

Lemma 2.E.1. *There are continuous functions $C_{r,k}$ so that if $u^\nu = (u^\nu_{-1}, ..., u^\nu_{r-2})$ satisfies the approximate system (2.E.25)-(2.E.27) and if the equation of state satisfies (2.2.18), then:*

$$||u^\nu_{-1}||_{H^r} \le C_{r,-1}(E_0, ||u_0^{\nu-1}||_{H^{r-2}})\kappa ||u_1^{\nu-1}||_{H^{r-2}}, \tag{2.E.35}$$

$$||u^\nu_k||_{H^{r-k}} \le C_{r,k}(E_0, ||u^{\nu-1}||_r, \textstyle\sum_{\ell \le k-1} ||u^\nu_\ell||_{H^{r-\ell-1}}) \big(\sum_{\ell \le k-1} ||u^\nu_\ell||_{H^{r-k}} + \kappa ||u^{\nu-1}||_r + \varepsilon\big), \tag{2.E.36}$$

and there are continuous functions $D_{r,k}$ so that with $U^\nu_k = u^\nu_k - u^{\nu-1}_k$:

$$||U^\nu_{-1}||_{H^r} \le D_{r,-1}(E_0, ||u_0^{\nu-1}||_r, ||u_0^{\nu-2}||_r)\kappa ||U_1^{\nu-1}||_{H^{r-2}}, \tag{2.E.37}$$

$$||U^\nu_k||_{H^{r-k}} \le D_{r,k}(E_0, ||u^\nu||_r, ||u^{\nu-1}||_r, ||u^{\nu-2}||_r) \big(\textstyle\sum_{\ell \le k-1} ||U^\nu_\ell||_{H^{r-\ell}} + \kappa ||U_\ell^{\nu-1}||_{H^{r-\ell}}\big). \tag{2.E.38}$$

Proof. Using the elliptic estimates (2.E.34) and the fact that $H^{r-k-2}(\Omega)$ is an algebra for $r - k \ge 4$, we have:

$$||\partial_{x^0} u^\nu_{-1}||_{H^r} \le C\big(||e'(h_0 + u_0^{\nu-1})||_{H^{r-1}} ||u_1^{\nu-1}||_{H^{r-1}}\big), \tag{2.E.39}$$

$$||\partial_{x^0} u_k^\nu||_{H^{r-k-1}} \leq C\big(||F_k - F_k^\nu||_{H^{r-k-2}} + ||G_k - G_k^{\nu-1}||_{H^{r-k-2}}$$

$$+ ||e'(h_0 + u_0^{\nu-1})||_{H^{r-k-2}}||u_{k+2}^{\nu-1}||_{H^{r-k-2}}\big), \quad (2.E.40)$$

for $k = 0, ..., r-2$, with the convention that $u_\ell^\nu = 0$ for $\ell \geq r-1$, and with constants depending on $||x_0||_{H^{r+1}}$.

Because $U^\nu = u^\nu - u^{\nu-1}$ satisfies the following system in Ω:

$$\Delta U_{-1}^\nu = e'(h_0 + u_0^{\nu-1})u_1^{\nu-1} - e'(h_0 + u_0^{\nu-2})u_1^{\nu-2}, \quad (2.E.41)$$

$$\Delta U_k^\nu = F_k^{\nu-1} - F_k^\nu + G_k^{\nu-1} - G_k^\nu + e'(h_0 + u_0^{\nu-1})u_{k+2}^{\nu-1}$$

$$- e'(h_0 + u_0^{\nu-2})u_{k+2}^{\nu-2}, \quad k = 0, ..., r-2, \quad (2.E.42)$$

with $U^\nu = 0$ on $\partial\Omega$, we also have:

$$||\partial_{x_0} U_{-1}^\nu||_{H^r} \leq C\big(||e'(h_0 + u_0^{\nu-1}) - e'(h_0 + u_0^{\nu-2})||_{H^{r-1}}||u_1^{\nu-1}||_{H^{r-1}}$$

$$+ ||e'(h_0 + u_0^{\nu-2})||_{H^{r-1}}||U_1^{\nu-1}||_{H^{r-1}}\big), \quad (2.E.43)$$

$$||U_k^\nu||_{H^{r-k}} \leq C\big(||F_k^{\nu-1} - F_k^\nu||_{H^{r-k-2}} + ||G_k^{\nu-1} - G_k^\nu||_{H^{r-k-2}}$$

$$+ ||e'(h_0 + u_0^{\nu-1}) - e'(h_0 + u_0^{\nu-2})||_{H^{s-k-2}}||u_{k+1}^{\nu-1}||_{H^{r-k-2}}$$

$$+ ||e'(h_0 + u_0^{\nu-2})||_{H^{r-k-2}}||U_{k+1}^{\nu-1}||_{H^{r-k-2}}. \quad (2.E.44)$$

The estimates (2.E.35)-(2.E.38) then follow from Proposition 2.E.2 and Lemmas 2.E.2-2.E.3. $\square$

It remains to prove estimates for the terms on the right-hand sides of (2.E.35),(2.E.36) and (2.E.37),(2.E.38). The proposition below is a consequence of Lemmas 2.E.5, 2.E.6, whose proofs we postpone until Section 2.E.1

Proposition 2.E.2. *Set* $M_k^\nu = ||\partial_{x_0} u_{-1}^\nu||_{H^r} + \sum_{j\leq k} ||u_j^\nu||_{H^{r-j}}$. *There are continuous functions* $K_k = K_k(E_0, M_{k-1}^\nu), K_k' = K_k'(E_0, M_{k-1}^\nu, M_{k-1}^{\nu-1})$ *so that writing* $j = r-k-2$*:*

$$||F_k - F_k^\nu||_{H^j} \leq K_k\big(||u_{-1}^\nu||_{H^{j+2}} + \textstyle\sum_{\ell\leq k-1} ||u_\ell^\nu||_{H^{j+2}} + \varepsilon\big), \tag{2.E.45}$$

$$||F_k^\nu - F_k^{\nu-1}||_{H^j} \leq K_k'\big(||U_{-1}^\nu||_{H^{j+2}} + \textstyle\sum_{\ell\leq k-1} ||U_\ell^\nu||_{H^{j+2}}\big). \tag{2.E.46}$$

Lemma 2.E.2. *There are continuous functions*

$$K = K(E_0, ||u^{\nu-1}||_{r-1})$$

,

$$K' = K'(E_0, ||u^{\nu-1}||_r, ||u^{\nu-2}||_r)$$

so that if $\sup_{r'\leq r+1} |e^{(r')}| \leq \kappa$ *then:*

$$||G_k - G_k^{\nu-1}||_{H^{r-k-2}} \leq \kappa K ||u^{\nu-1}||_r, \quad ||G_k^{\nu-1} - G_k^{\nu-2}||_{H^{r-k-2}} \leq \kappa K' ||u^{\nu-1} - u^{\nu-2}||_r \tag{2.E.47}$$

Proof. Write $h_k^\nu = h_k + u_k^{\nu-1}$. Expanding out the definition of G_k, G_k^ν and applying ∂_y^I for a multi-index I with $|I| = r' \leq r-k-2$, we see that $\partial_y^I(G_k - G_k^\nu)$ is a sum of terms of the form:

$$e^{(K)}(h_0^{\nu-1})(\partial_y^{J_1} h_{k_1}^{\nu-1}) \cdots (\partial_y^{J_j} h_{k_j}^{\nu-1}) - e^{(K)}(h_0)(\partial_y^{J_1} h_{k_1}) \cdots (\partial_y^{J_j} h_{k_j}), \tag{2.E.48}$$

with $|J_1| + \cdots |J_j| = r-k-2, k_1 + \cdots k_j = k+1, K \leq r-1$. Performing the usual manipulations, rearranging terms, and using that $h_k^{\nu-1} - h_k = u_k^{\nu-1}$, it suffices to control the $L^2(\Omega)$ of a sum of terms of the form:

$$(e^{(K)}(h_0^{\nu-1}) - e^{(K)}(h_0))(\partial_y^{J_1} h_{k_1}^{\nu-1}) \cdots (\partial_y^{J_j} h_{k_j}^{\nu-1}), \tag{2.E.49}$$

and

$$e^{(K)}(h_0^{\nu-1})(\partial_y^{J_1} u_{k_1}^{\nu-1})(\partial_y^{J_2} h_{k_2}^{\nu-1}) \cdots (\partial_y^{J_j} h_{k_j}^{\nu-1}), \tag{2.E.50}$$

the remaining terms being similar but with some of the factors of $h_\ell^{\nu-1}$ replaced by h_ℓ. Let us just bound the second type of term here, the first type being identical after using the estimate $|e^{(K)}(h_0^{\nu-1}) - e^{(K)}(h_0)| \leq |\sup e^{(K+1)}||u_0^{\nu-1}|$. For each ℓ with $|J_\ell| + k_\ell \leq r-3$, we bound the resulting term in L^∞ by Sobolev embedding to get either $||\partial_y^{J_\ell} u_{k_\ell}^{\nu-1}||_{L^\infty(\Omega)} \leq C||\partial_y^{J_\ell} u_{k_\ell}^{\nu-1}||_{H^2(\Omega)}$ or $C(||\partial_y^{J_\ell} h_{k_\ell}||_{H^2(\Omega)} + ||\partial_y^{J_\ell} u_{k_\ell}^{\nu-1}||_{H^2(\Omega)})$. Since $|J_\ell| + k_\ell + 2 \leq r-1$, the result can be bounded by $||u^{\nu-1}||_{s-1}$ or $||u^{\nu-1}||_{r-1} + E_0$, respectively. It therefore remains to handle terms with $|J_\ell| + k_\ell \geq r-2$. Since $r \geq 5$ there is at most one such term and so it is bounded by either $||u_{k_\ell}^{\nu-1}||_{H^{|J_\ell|}(\Omega)} \leq ||u^{\nu-1}||_{r-1}$ or $||h_{k_\ell}||_{H^{|J_\ell|}(\Omega)} + ||u_{k_\ell}^{\nu-1}||_{H^{|J_\ell|}(\Omega)} \leq E_0 + ||u^{\nu-1}||_{r-1}$, as required. The estimate for $G^{\nu-1} - G^{\nu-2}$ is similar. □

Lemma 2.E.3. *There are continuous functions $K'' = K''(E_0, ||u_0^{\nu-1}||_{H^{r-1}})$, $K''' = K'''(E_0, ||u_0^{\nu-1}||_{H^{r-1}}, ||u_0^{\nu-2}||_{H^{r-1}})$ so that if $\sup_{k \leq r+1} |e^{(k)}| \leq \kappa$ then:*

$$||e'(h_0^{\nu-1})||_{H^r} \leq \kappa K''||u_0^{\nu-1}||_{H^r(\Omega)}, \tag{2.E.51}$$

$$||e'(h_0^{\nu-1}) - e'(h_0^{\nu-2})||_{H^r} \leq \kappa K'''||u_0^{\nu-1} - u_0^{\nu-2}||_{H^r}. \tag{2.E.52}$$

Proof. By the chain rule, if I is a multi-index with $|I| = r' \leq r$, $\partial_y^I(e'(h_0 + u_0^{\nu-1}))$ is a sum of terms of the form:

$$e^{(K)}(h_0 + u_0^{\nu-1})(\partial_y^{J_1} h_0 + u_0^{\nu-1}) \cdots (\partial_y^{J_j} h_0 + u_0^{\nu-1}), \qquad \sum |J_j| = r', K \leq r'. \tag{2.E.53}$$

We want to control the $L^2(\Omega)$ norm of this. For each ℓ with $|J_\ell| \leq r-3$ we control the L^∞ norm of the resulting factor by Sobolev embedding which shows that any such term is bounded by $C(||h_0||_{H^{r-1}(\Omega)} + ||u_0^{\nu-1}||_{H^{r-1}(\Omega)})$. To handle terms with $|J_\ell| \geq r-2$, note that since $r \geq 5$ there can be at most one such term and we control it by $||u_0^{\nu-1}||_{H^r(\Omega)}$. Since $|e^{(K)}| \leq \kappa$ this gives the first estimate (2.E.52) and the second is similar. □

2.E.1 Estimates for $F_k - F_k^\nu$ and $F_k^\nu - F_k^{\nu-1}$

For these estimates it will be convenient to first state the results in terms of the coefficients V_k, V_k^ν before relating these to h_k, u_k^ν, because they depend on each other in a complicated way. Recall the definitions of $S, \tilde{S}$ from (2.2.21), (2.4.17). Given power series in time t $\hat{V}, \hat{V}_\varepsilon$ as in the beginning of this section and evaluating at $t=0$, the $S, \widetilde{S}$ are polynomials in the following arguments:

$$S_\ell^k = S_\ell^k(\partial V_0, ..., \partial V_{k-\ell-1}), \qquad \widetilde{S}_\ell^k = \widetilde{S}_\ell^k(\partial S_\varepsilon V_0^\varepsilon, ..., \partial S_\varepsilon V_{k-\ell-1}^\varepsilon). \tag{2.E.54}$$

We note for later use that in fact we have:

$$S_\ell^k(\partial V_0, ..., \partial V_{k-\ell-1}) = \widetilde{S}_\ell^k(\partial V_0, ..., \partial V_{k-\ell-1}), \tag{2.E.55}$$

which follows from the formulas (2.2.21), (2.4.17). We then have the following formula for the V_k:

$$V_k^i = -\delta^{ii'}\partial_{i'} H_{k-1} + \sum\nolimits_{\ell \leq k-2} \delta^{ii'} S_{i'\ell}^{jk} \partial_j H_\ell, \qquad i = 1, 2, 3, \tag{2.E.56}$$

and, given V_0^ν we recursively define V_k^ν by:

$$V_k^{i\nu} = -\delta^{ii'}\partial_{i'} H_{k-1}^\nu + \sum\nolimits_{\ell \leq k-2} \delta^{ii'} \widetilde{S}_{i'\ell}^{jk,\nu} \partial_j H_\ell^\varepsilon, \qquad i = 1, 2, 3, \tag{2.E.57}$$

where, with $\tilde{S}_{i\ell}^{jk}$ defined in (2.4.17), we are writing:

$$\widetilde{S}_{i\ell}^{jk,\nu} = \widetilde{S}_{i\ell}^{jk}(\partial S_\varepsilon V_0^\nu, ..., \partial S_\varepsilon V_{k-\ell-1}^\nu). \tag{2.E.58}$$

We are also writing $H_\ell = h_\ell + \phi_\ell$ and $H^\nu_\ell = h^\nu_\ell + \phi^\nu_\ell$ with $\phi^\nu_\ell = D_t^\ell \phi[x, \hat{h}^\nu]|_{t=0}$, where $\hat{h}^\nu(t) = \sum h^\nu t^k/k!$.

If T is a (2,2) tensor then we write:

$$||T^k_\ell||^2_{H^m} = \sum_{0\le |I|\le m} \int_\Omega \delta^{ii'}\delta_{jj'}(\partial^I_y T^{jk}_{i\ell})(\partial^I_y T^{j'k}_{i'\ell})dy, \tag{2.E.59}$$

and we have the following lemma which will be used repeatedly to control the commutators $S, \widetilde{S}$:

Lemma 2.E.4. *Let $e^s_0 = E_0 + \sum_{j\le s}||V_j||_{H^{s-j}}$. If $s \geq 2$ then there are continuous functions $C_{k,\ell}$ so that*

$$||S^k_\ell||_{H^s} \le C_{k,\ell}(e^{s+1}_0), \qquad ||\widetilde{S}^{k,\nu}_\ell||_{H^s} \le C_{k,\ell}(e^{s+1}_0, m^\nu_{k-\ell+1,s+1}), \tag{2.E.60}$$

where

$$m^\nu_{r,s+1} = \sum_{j\le r}||V^\nu_j||_{H^{s+1}}, \tag{2.E.61}$$

and there are continuous functions $D_{k,\ell}, D'_{k,\ell}$ so that:

$$||S^k_\ell - \widetilde{S}^k_\ell||_{H^s} \le D_{k,\ell}(e^{s+1}_0, m^\nu_{k-\ell+1,s+1})\left(\sum_{j\le k-\ell-1}||V_j - V^{\nu-1}_j||_{H^{m+1}} + \varepsilon e^{s+2}_0\right), \tag{2.E.62}$$

$$||\widetilde{S}^{k,\nu}_\ell - \widetilde{S}^{k,\nu-1}_\ell||_{H^s} \le D'_{k,\ell}(e^{s+1}_0, m^\nu_{k-\ell+1,s+1}, m^{\nu-1}_{k-\ell+1,s+1})\sum_{j\le k-\ell-1}||V^\nu_j - V^{\nu-1}_j||_{H^{s+1}}. \tag{2.E.63}$$

Proof. Because $s \geq 2$, $H^s(\Omega)$ is an algebra and so the first two estimates follow because $S, \widetilde{S}$ are polynomials in their arguments (see (2.2.21) and (2.4.17)).

To prove (2.E.62), set $A = (\partial V_0, ..., \partial V_{k-\ell-1})$ and $\widetilde{A}^\nu = (\partial S_\varepsilon V^\nu_0, ..., \partial S_\varepsilon V^\nu_{k-\ell-1})$. Abusing notation, we write:

$$S^k_\ell - \widetilde{S}^{k,\nu}_\ell = S^k_\ell(A) - \widetilde{S}^k_\ell(\widetilde{A}^\nu) = S^k_\ell(A) - \widetilde{S}^k_\ell(A) + (\widetilde{S}^k_\ell(\widetilde{A}^\nu) - \widetilde{S}^k_\ell(A)). \tag{2.E.64}$$

By (2.E.55), the first two terms cancel. Since $\widetilde{S}$ is a polynomial in its arguments, we have:

$$||\widetilde{S}^k_\ell(\widetilde{A}^\nu) - \widetilde{S}^k_\ell(A)||_{H^s} \leq C' \sum_{j \leq k-\ell-1} ||V_j - S_\varepsilon V^\nu_j||_{H^{s+1}}, \tag{2.E.65}$$

with $C = C(E^s_0, ||A||_{H^s}, ||A^\nu||_{H^s})$, after additionally using that S_ε is bounded on Sobolev spaces. Now we write $||V_j - S_\varepsilon V^\nu_j||_{H^{s+1}} \leq ||V_j - S_\varepsilon V_j||_{H^{s+1}} + ||S_\varepsilon(V_j - V^\nu_j)||_{H^{s+1}}$. Since $||V_j - S_\varepsilon V_j||_{H^{s+1}} \leq C\varepsilon ||V_j||_{H^{s+2}}$ by (2.A.26), this concludes the proof of the third estimate. The proof of (2.E.63) is similar. □

We have the following technical estimate for $F_k - F^\nu_k$ and $F^\nu_k - F^{\nu-1}_k$ in terms of V_k, V^ν_k:

Lemma 2.E.5. *Set $m^\nu_k = ||V^\nu_0||_{H^r} + \sum_{0 \leq j \leq k} ||V^\nu_j||_{H^{r-j-1}}$. There are continuous functions $K = K_{r,k}(E_0, m^\nu_k), K' = K'_{r,k}(E_0, m^\nu_k, m^{\nu-1}_k)$ so that, with V_k, V^ν_k defined by (2.E.56)-(2.E.57) and $j = r - k - 2$:*

$$||F_k - F^\nu_k||_{H^j} \leq K\big(||u^\nu_{-1}||_{H^{j+2}} + \sum_{\ell \leq k} ||V_\ell - V^\nu_\ell||_{H^{j+1}} + ||u^\nu_{\ell-1}||_{H^{j+2}} + \varepsilon\big), \tag{2.E.66}$$

$$\begin{aligned} ||F^\nu_k - F^{\nu-1}_k||_{H^j} \leq K'\big(&||u^\nu_{-1} - u^{\nu-1}_{-1}||_{H^{j+2}} \\ &+ \sum_{\ell \leq k} ||V^\nu_\ell - V^{\nu-1}_\ell||_{H^{j+1}} + ||u^\nu_\ell - u^{\nu-1}_\ell||_{H^{j+2}}\big). \end{aligned} \tag{2.E.67}$$

Proof. We start by writing $F_k - F^\nu_k$ more explicitly in terms of the S and $\widetilde{S}$. With V^ν_k defined in (2.E.57) and with $h^\nu_k = h_k + u^\nu_k$, let $\hat{V}^\nu(t) = \sum_{k=0}^N V^\nu_k t^k/k!$ and $\hat{h}^\nu(t) = \sum_{k=0}^N h^\nu_k t^k/k!$, we write:

$$\begin{aligned} F_k - F^\nu_k = D^k_t\big((\hat{\partial}\hat{V})(\hat{\partial}\hat{V}) - (\hat{\tilde{\partial}} S_\varepsilon \hat{V}^\nu)(\hat{\tilde{\partial}}\hat{V}^\nu)\big)|_{t=0} + \big([D^k_t, \hat{\Delta}]\hat{h} - [D^k_t, \hat{\tilde{\Delta}}]\hat{h}^\nu\big)|_{t=0} \\ \equiv f^\nu_k + g^\nu_k. \end{aligned} \tag{2.E.68}$$

For matrices a^j_i, b^j_i, write $a \cdot b = a^j_i b^i_j$ and if T is a (2,2) tensor, write $T^k_\ell a$ for the matrix with components $(T^k_\ell a)^n_m = T^{jk}_{m\ell} a^n_j$. We then have the following expression:

$$f^\nu_k = \sum_{k_1+k_2=k} \sum_{\ell \le k_1} \sum_{\ell' \le k_2} S^{k_1}_\ell \partial V_\ell \cdot S^{k_2}_{\ell'} \partial V_{\ell'} - \widetilde{S}^{k_1,\nu}_\ell \partial \widetilde{V}^\nu_\ell \cdot \widetilde{S}^{k_2,\nu}_{\ell'} \partial V^\nu_{\ell'}, \qquad \widetilde{V}^\nu_k = S_\varepsilon V^\nu_k. \quad (2.E.69)$$

Using the commutator formulas (2.2.20),(2.4.17) twice, we have:

$$g^\nu_k = \sum_{\ell \le k-1} \delta^{ij} \big(\partial_i S^{i'k}_{j\ell} \partial_{i'} h_\ell - \partial_i \widetilde{S}^{i'k,\nu}_{j\ell} \partial h^\nu_\ell + S^{i'k}_{j\ell} \partial^2_{ii'} h_\ell - \widetilde{S}^{i'k,\nu}_{j\ell} \partial^2_{ii'} h^\nu_\ell \big)$$

$$+ \sum_{\ell \le k-1} \sum_{\ell' \le \ell-1} \delta^{ij} \big(S^{i'k}_{i\ell} \partial_{i'} S^{j'\ell}_{j\ell'} \partial_{j'} h_{\ell'} - \widetilde{S}^{i'k,\nu}_{i\ell} \partial_{i'} \widetilde{S}^{j'\ell,\nu}_{j\ell'} \partial_{j'} h^\nu_{\ell'}$$

$$+ S^{i'k}_{i\ell} S^{j'\ell}_{j\ell} \partial^2_{i'j'} h_{\ell'} - \widetilde{S}^{i'k,\nu}_{i\ell} \widetilde{S}^{j'\ell,\nu}_{j\ell} \partial^2_{i'j'} h^\nu_{\ell'} \big), \quad (2.E.70)$$

where here we are writing $\partial = \partial_{x_0}$. Similarly, we have $F^\nu_k - F^{\nu-1}_k = f^{\nu,\nu-1}_k + g^{\nu,\nu-1}_k$, where:

$$f^{\nu,\nu-1}_k = f^\nu_k - f^{\nu-1}_k$$

$$= \sum_{k_1+k_2=k} \sum_{\ell \le k_1} \sum_{\ell' \le k_2} \widetilde{S}^{k_1,\nu}_\ell \partial \widetilde{V}^\nu_\ell \cdot \widetilde{S}^{k_2,\nu}_{\ell'} \partial V^\nu_{\ell'} - \widetilde{S}^{k_1,\nu-1}_\ell \partial \widetilde{V}^{\nu-1}_\ell \cdot \widetilde{S}^{k_2,\nu-1}_{\ell'} \partial V^{\nu-1}_{\ell'}, \quad (2.E.71)$$

and

$$g^{\nu,\nu-1}_k = g^\nu_k - g^{\nu-1}_k = \sum_{\ell \le k-1} \partial \widetilde{S}^{k,\nu}_\ell \partial h^\nu_\ell - \partial \widetilde{S}^{k,\nu-1}_\ell \partial h^{\nu-1}_\ell + \widetilde{S}^{k,\nu}_\ell \partial^2 h^\nu_\ell - \widetilde{S}^{k,\nu-1}_\ell \partial^2 h^{\nu-1}_\ell$$

$$+ \sum_{\ell \le k-1} \sum_{\ell' \le \ell} \widetilde{S}^{\ell,\nu}_{k-1} \partial \widetilde{S}^{\ell',\nu}_\ell \partial h^\nu_{\ell'} - \widetilde{S}^{\ell,\nu-1}_{k-1} \partial \widetilde{S}^{\ell',\nu-1}_\ell \partial h^{\nu-1}_{\ell'} + \widetilde{S}^{\ell,\nu}_{k-1} \widetilde{S}^{\ell',\nu}_\ell \partial^2 h^\nu_{\ell'} - \widetilde{S}^{\ell,\nu-1}_{k-1} \widetilde{S}^{\ell',\nu-1}_\ell \partial^2 h^{\nu-1}_{\ell'}. \quad (2.E.72)$$

We first consider the case $r - k - 2 \ge 2$. After performing the usual manipulations and using that H^{r-k-2} is an algebra, to control $||f^\nu_k||_{H^{r-k-2}}$, it suffices to prove that for $k' \le k, \ell \le k$, writing $j = r - k - 2$

$$||S^{k'}_\ell||_{H^j} + ||\widetilde{S}^{k',\nu}_\ell||_{H^j} + ||\partial V_\ell||_{H^j} + \sum_{\alpha=0,1} ||\partial S^\alpha_\varepsilon V_\ell||_{H^j} \le K(E_0, m^\nu_k), \quad (2.E.73)$$

and that, with $K' = K'(E_0, m_k^\nu, m_k^{\nu-1})$:

$$||S_\ell^{k'} - \widetilde{S}_\ell^{k',\nu}||_{H^j} + ||\partial V_\ell - \partial V_\ell^\nu||_{H^j} + ||\partial V_\ell - \partial \widetilde{V}_\ell^\nu||_{H^j}$$

$$\leq K'\big(\textstyle\sum_{\ell \leq k}||V_\ell - V_\ell^\nu||_{H^{j+1}} + ||u_{-1}^\nu||_{H^{j+2}} + \varepsilon E_0\big). \quad (2.E.74)$$

The first two terms in (2.E.73) are bounded by the right side of (2.E.73) by Lemma 2.E.4 and the other terms are bounded by the right side using the definition of the m_k^ν and the fact that S_ε is bounded on Sobolev spaces.

The first term in (2.E.74) is bounded by the right-hand side of (2.E.74) by Lemma 2.E.4, and the second term is directly bounded by $||V_\ell - V_\ell^\nu||_{H^{r-k-1}}$. To control the third term, we write $\widetilde{V}_\ell^\nu = S_\varepsilon V_\ell^\nu = S_\varepsilon V_\ell + S_\varepsilon(V_\ell^\nu - V_\ell)$. Using that $||(1 - S_\varepsilon)\partial V_\ell||_{H^{r-k-2}} \leq C\varepsilon||V_\ell||_{H^{r-k}}$ and $||S_\varepsilon(V_\ell - V_\ell^\nu)||_{H^{s-k-1}} \leq C||V_\ell - V_\ell^\nu||_{H^{r-k-1}}$ gives the bound for f_k^ν. The bound for $f_k^{\nu,\nu-1}$ follows in a nearly identical way. The case $r - k - 2 \leq 1$ is similar and follows the same lines as the proof of e.g. (2.D.9).

We now bound g_k^ν. We just prove estimates for the terms on the first line of (2.E.72) as the terms on the second line can be bounded in a similar manner. It suffices to prove that for $\ell \leq k-1$ with $j = r - k - 2$:

$$\textstyle\sum_{m=0,1}||\partial^m S_\ell^k||_{H^j} + ||\partial^m \widetilde{S}_\ell^{k,\nu}||_{H^j} + ||\partial^{m+1}h_\ell||_{H^j} + ||\partial^{m+1}h_\ell^\nu||_{H^j} \leq K(E_0, m_k^\nu), \quad (2.E.75)$$

$$\textstyle\sum_{m=0,1}||\partial^m S_\ell^k - \partial^m \widetilde{S}_\ell^{k,\nu}||_{H^j} + ||\partial^{m+1}h_\ell - \partial^{m+1}h_\ell^\nu||_{H^j}$$

$$\leq K'\big(\textstyle\sum_{\ell' \leq k}||V_{\ell'} - V_{\ell'}^\nu||_{H^{j+1}} + ||u_k^\nu||_{H^{j+2}} + ||u_{-1}^\nu||_{H^{j+2}}\big). \quad (2.E.76)$$

These estimates follow from Lemma 2.E.4 since $h_k^\nu = h_k + u_k^\nu$. The estimates for $g_k^\nu - g_k^{\nu-1}$ are similar. $\square$

To complete the proof of the estimates for $F_k - F_k^\nu$, we need the following two estimates to relate V_k, V_k^ν to the initial data V_0, h_0 and the perturbations u^ν. We need a bit more notation. Given a diffeomorphism $X : \Omega \to X(\Omega)$ and a function $f : \Omega \to \mathbb{R}$, let $\Phi[X, f] = \phi \circ X^{-1}$, where ϕ is defined by:

$$(X, f) \mapsto \phi(x) = \int_{X(\Omega)} |x - x'|^{-1} \rho(f(x'))\, dx', \quad x \in \mathbb{R}^3. \tag{2.E.77}$$

Set $\hat{x} = x_0 + t \sum_{k\geq 0} V_k t^k/(k+1)!$, $\hat{x}^\nu(t) = x_0 + t\sum_{k\geq 0} V_k^\nu t^k/(k+1)!$ and write $x_\ell = D_t^\ell \hat{x}|_{t=0}$, $x_\ell^\nu = D_t^\ell \hat{x}^\nu|_{t=0}$. Set $\phi_\ell = D_t^\ell \Phi[\hat{x}, \hat{h}]|_{t=0}$ and $\phi_\ell^\nu = D_t^\ell \Phi[\hat{x}^\nu, \hat{h}^\nu]|_{t=0}$. Then:

Lemma 2.E.6. *With notation as in the previous lemma, for each $k = -1, ..., r-2$, there are continuous functions $K_0 = K_0(E_0)$, $K_0' = K_0'(E_0, ||u_{-1}^\nu||_{H^{r-k+1}}, \sum_{\ell \leq k-1} ||u_\ell^\nu||_{H^{r-k+1}})$, $K_0'' = K_0''(E_0, ||u^\nu||_r, ||u^{\nu-1}||_r)$ so that:*

$$||V_k||_{H^{r-k}} \leq K_0, \quad ||V_k^\nu||_{H^{r-k}} \leq K_0', \tag{2.E.78}$$

$$||V_k - V_k^\nu||_{H^{r-k}} \leq K_0' \big(||u_{-1}^\nu||_{H^{r-k+1}} + \textstyle\sum_{\ell \leq k-1} ||u_\ell^\nu||_{H^{r-k+1}}\big), \tag{2.E.79}$$

and with $U^\nu = u^\nu - u^{\nu-1}$:

$$||V_k^\nu - V_k^{\nu-1}||_{H^{r-k}} \leq K_0''\big(||U_{-1}^\nu||_{H^{r-k+1}} + \textstyle\sum_{\ell\leq k-1} ||U_\ell^\nu||_{H^{r-k+1}}\big). \tag{2.E.80}$$

Proof. These estimates all follow from the definitions of V_k, V_k^ν in (2.E.56),(2.E.57), the estimates for $S, \widetilde{S}$ from Lemma 2.E.4, and the estimates for $\phi_{k-1}, \phi_{k-1}^\nu$ in Lemma 2.E.7. $\square$

It still remains to control $\phi_{k-1}, \phi_{k-1}^\nu$. We shall not use this observation to prove estimates, but we remark that we have the following explicit representation formula for ϕ_{k-1}:

$$\phi_{k-1}(y) = \textstyle\sum_{k_1 + \dots k_j + k' \leq k-1} \displaystyle\int_\Omega K_{k_1, \dots, k_j}(y, y') J_{k'}(y')\, dy', \tag{2.E.81}$$

where, for some constants $d_{k_1 \cdots k_j}$:

$$K_{k_1,\dots,k_j}(y,y') = d_{k_1..k_j} \frac{(\delta x_{k_1} \cdot \delta x_{k_2}) \cdots (\delta x_{k_{j-1}} \cdot \delta x_{k_j})}{|x_0(y) - x_0(y')|}, \qquad J_{k'} = D_t^{k'}(\rho(\hat{h})\hat{\kappa})|_{t=0}, \tag{2.E.82}$$

with $\hat{\kappa} = \det(\partial y/\partial \hat{x})$, and where we are writing $\delta W(y,y) = (W(y) - W(y'))/|x_0(y) - x_0(y')|$.

Similarly:

$$\phi^{\nu}_{k-1}(y) = \sum_{k_1+\cdots k_j+k' \le k-1} \int_{\Omega} K^{\nu}_{k_1,\dots,k_j}(y,y') J^{\nu}_{k'}(y')\, dy', \tag{2.E.83}$$

where

$$K^{\nu}_{k_1,\dots,k_j}(y,y') = d_{k_1..k_j} \frac{(\delta x^{\nu}_{k_1} \cdot \delta x^{\nu}_{k_2}) \cdots (\delta x^{\nu}_{k_{j-1}} \cdot \delta x^{\nu}_{k_j})}{|x_0(y) - x_0(y')|}, \tag{2.E.84}$$

$$J^{\nu}_{k'} = D_t^{k'}(\rho(\hat{h^{\nu}})\hat{\kappa^{\nu}})|_{t=0}, \qquad \widehat{\kappa}^{\nu} = \det(\partial \hat{x}^{\nu}/\partial y). \tag{2.E.85}$$

Lemma 2.E.7. *With notation as in Lemma 2.E.5, for each $k = 0, ..., r-2$, there are continuous functions $K_0 = K_0(E_0)$, $K_0' = K_0'(E_0, m^{\nu}_{k-1})$, $K_0'' = K_0''(E_0, ||u^{\nu}||_s, ||u^{\nu-1}||_s)$ so that:*

$$||\phi_{k-1}||_{H^{r-k+1}} \le K_0, \qquad ||\phi^{\nu}_{k-1}||_{H^{r-k+1}} \le K_0'. \tag{2.E.86}$$

$$||\phi_{k-1} - \phi^{\nu}_{k-1}||_{H^{r-k+1}} \le K_0' \sum_{\ell \le k-1} ||u^{\nu}_{\ell}||_{H^{r-k}}, \tag{2.E.87}$$

$$||\phi^{\nu}_{k-1} - \phi^{\nu-1}_{k-1}||_{H^{r-k+1}} \le K_0'' \sum_{\ell \le k-1} ||U^{\nu}_{\ell}||_{H^{r-k}}. \tag{2.E.88}$$

Proof. The estimates follow from Theorem 2.7.4, respectively Theorem 2.7.7. □

2.F Existence for the wave equations

2.F.1 Existence for the linear wave equations

Fixing $r \geq 7, T > 0, V \in \mathcal{X}^{r+1}(T)$ and defining $\widetilde{\Delta} = \widetilde{\Delta}[V]$ as in (2.4.4), the goal of this section is to solve the linear wave equation:

$$D_t^2 \varphi - \sigma \widetilde{\Delta} \varphi = \mathcal{F}, \quad \text{in } [0,T] \times \Omega, \quad \text{with} \quad \varphi = 0, \text{ on } [0,T] \times \partial\Omega, \tag{2.F.1}$$

$$\varphi(0,y) = \varphi_0(y), \quad D_t \varphi(0,y) = \varphi_1(y), \quad \text{in } \Omega, \tag{2.F.2}$$

where $\sigma = \sigma(t,y)$ satisfies $0 < c_0 < \sigma \leq c_1$ for some constants c_0, c_1. We will omit the dependence on c_0, c_1 in what follows. We remark that compared with the estimates in Section 2.6, we have divided by σ and abused notation slightly to make the following computations simpler.

As in Section 2.4.3, there are compatibility conditions for (2.F.1)-(2.F.2).We define φ_k for $k \geq 2$ recursively by:

$$\varphi_k = \big(D_t^{k-2}(\sigma \widetilde{\Delta} \varphi) + D_t^{k-2} \mathcal{F} \big)\big|_{t=0}. \tag{2.F.3}$$

We say that φ_0, φ_1 satisfy the compatibility conditions to order s if:

$$\varphi_k \in H_0^1(\Omega), \qquad k = 0, ..., s. \tag{2.F.4}$$

We will control solutions to (2.F.1)-(2.F.2) using the quantities:

$$Y_s(t) = \Big(\sum_{k \leq s} \int_\Omega |D_t^{k+1} \varphi|^2 + \sigma \delta^{ij} (D_t^k \widetilde{\partial}_i \varphi)(D_t^k \widetilde{\partial}_j \varphi) \, dy \Big)^{1/2}. \tag{2.F.5}$$

The main result we need is:

Proposition 2.F.1. *Fix $r \geq 7$ and $T \geq 0$, and suppose that $V \in \mathcal{X}^{r+1}(T)$ satisfies (2.9.12). Suppose* *also that:*

$$\widetilde{x} \in L^\infty([0,T]; H^r(\Omega)), \tag{2.F.6}$$

$$D_t\widetilde{x} \in L^\infty([0,T]; H^r(\Omega)), \quad \textit{and} \quad D_t^k D_t\widetilde{x} \in L^\infty([0,T]; H^{r-k+1}(\Omega)), \quad k = 1, ..., r+1, \tag{2.F.7}$$

$$D_t^k \sigma \in L^\infty([0,T]; H^{r-k}(\Omega)), \qquad k = 0, ..., r, \tag{2.F.8}$$

and that the bound (2.5.2) *holds. Assume also that for some* s *with* $0 \le s \le r$*,*

$$\mathcal{F} \in L^\infty([0,T]; H^{s-1}(\Omega)), \quad \textit{and} \quad D_t^k \mathcal{F} \in L^\infty([0,T]; H^{s-k}(\Omega)), \qquad k = 1, ..., s, \tag{2.F.9}$$

$$\varphi_k \in H_0^1(\Omega), \qquad\qquad k = 0, ..., s. \tag{2.F.10}$$

Take $K = K_{s,r}$ *so that*

$$\sup{}_{0 \le t \le T} \big(||\widetilde{x}(t)||_r + ||V(t)||_r + ||D_t V(t)||_r + ||\sigma(t)||_r + ||F(t)||_{s-1} \big) \le K. \tag{2.F.11}$$

Then the problem (2.F.1)-(2.F.2) *has a unique solution* φ *satisying:*

$$D_t^{s+1}\varphi \in L^\infty([0,T]; L^2(\Omega)), \qquad D_t^\ell \widetilde{\partial}\varphi \in L^\infty([0,T]; H^{s-\ell}(\Omega)), \qquad \ell = 0, ..., s, \tag{2.F.12}$$

and there are continuous functions $\mathcal{C}_s$ *depending on* $M, Y_{s-1}(0), T$*, and* K *so that:*

$$\sup{}_{0 \le t \le T} Y_s(t) \le \mathcal{C}_s \Big(Y_s(0) + \int_0^T ||\mathcal{F}(\tau)||_{s,0}\, d\tau \Big), \tag{2.F.13}$$

and for $0 \le t \le T$

$$||\widetilde{\partial}\varphi(t)||_s \le \mathcal{C}_s (Y_s(t) + ||\mathcal{F}(t)||_{s-1}), \tag{2.F.14}$$

$$||\widetilde{\partial}\varphi(t)||_r \le \mathcal{C}_r \big(Y_r(t) + ||\mathcal{F}(t)||_{r-1} + \varepsilon^{-1}(||J_\varepsilon x(t)||_r + 1) Y_{r-1}(t) \big). \tag{2.F.15}$$

This result is well-known (see e.g. [14] or [12]) and will follow from a Galerkin method.

However, we will need to be careful about the regularity of $\widetilde{x}$ and we will use our elliptic estimates from Section 2.B in place of "standard" elliptic estimates. We do not claim that this result is optimal with respect to the total number of derivatives of $\widetilde{x}, V, D_tV, \sigma$ required and in many of the following results it is obvious that one can do with much weaker assumptions on these variables. We start by constructing weak solutions to the system (2.F.1)-(2.F.2). Let $\{e_k\}_{k=0}^{\infty}$ be the L^2-normalized eigenfunctions in $H^1_0(\Omega)$ of the Dirichlet Laplacian in the y-coordinates $\Delta_y = \partial_1^2 + \partial_2^2 + \partial_3^2$. Let $d_m^k \in C^2([0,T])$, $k = 1, ..., m$ solve the following system:

$$D_t^2 d_m^k + B_k(d_m) = \int_\Omega \mathcal{F} e_k \, dy, \qquad k = 1, ..., m, \tag{2.F.16}$$

$$d_m^k(0) = (\varphi_0, e_k), \qquad D_t d_m^k(0) = (\varphi_1, e_k), \qquad k = 1, ..., m, \tag{2.F.17}$$

where:

$$B_k(d_m) = \textstyle\sum_{\ell=1}^k d_m^\ell \displaystyle\int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i e_\ell)(\widetilde{\partial}_j e_k) \, dy. \tag{2.F.18}$$

Define:

$$\varphi^m(t) = \textstyle\sum_{k=1}^m d_m^k(t) e_k. \tag{2.F.19}$$

Multipling (2.F.16) by d_m^k, summing over $k \leq m$ and using (2.F.19), we have:

$$\int_\Omega D_t^2 \varphi^m e_k \, dy + \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi^m)(\widetilde{\partial}_j e_k) \, dy = \int_\Omega \mathcal{F} e_k \, dy, \qquad k = 1, ..., m. \tag{2.F.20}$$

We now prove the basic energy estimate:

Lemma 2.F.1. *If φ^m is as above, there is a constant $C_0 = C_0(M, K)$ so that:*

$$\max_{0 \leq t \leq T} \big(||D_t \varphi^m(t)||_{L^2(\Omega)} + ||\widetilde{\partial} \varphi^m(t)||_{L^2(\Omega)} \big) + ||D_t^2 \varphi^m||_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leq C_0 \big(||\varphi_0||_{H^1(\Omega)} + ||\varphi_1||_{L^2(\Omega)} + ||\mathcal{F}||_{L^2(0,T;L^2(\Omega))} \big). \tag{2.F.21}$$

Proof. We multiply (2.F.20) by $D_t d_m^k$ and sum over $k = 1, ..., m$ to get:

$$\int_\Omega (D_t^2 \varphi^m)(D_t \varphi^m)\, dy + \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi^m)(\widetilde{\partial}_j D_t \varphi^m)\, dy = \int_\Omega \mathcal{F} D_t \varphi^m dy. \tag{2.F.22}$$

This first term is $2^{-1} d\, ||D_t \varphi^m||_{L^2(\Omega)}/dt$. We use (2.D.3) and write the second term as:

$$\int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi^m) D_t (\widetilde{\partial}_j \varphi^m) dy - \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_j V^\ell)(\widetilde{\partial}_i \varphi^m)(\widetilde{\partial}_\ell \varphi^m) dy$$

$$= \frac{1}{2}\frac{d}{dt} ||\sqrt{\sigma}\widetilde{\partial}\varphi^m||^2_{L^2(\Omega)} - \int_\Omega \delta^{ij} (\widetilde{\partial}_i \varphi^m)\big(\sigma(\widetilde{\partial}_j S_\varepsilon V^\ell)(\widetilde{\partial}_\ell \varphi) + D_t \sigma \widetilde{\partial}_j \varphi^m\big)\, dy, \tag{2.F.23}$$

and we can bound this last term by $C(M)(1 + ||D_t \sigma||_{L^\infty(\Omega)})||\widetilde{\partial}\varphi^m||^2_{L^2}$.

Writing $Y_{(m)}(t) = ||D_t \varphi^m||_{L^2} + ||\sqrt{\sigma}\widetilde{\partial}\varphi^m||_{L^2}$, we have shown:

$$\frac{1}{2}\frac{d}{dt} Y^2_{(m)} \leq C(M)\Big((1 + ||D_t \sigma||_{L^\infty(\Omega)}) Y^2_{(m)} + ||\mathcal{F}||_{L^2} Y_{(m)}\Big), \tag{2.F.24}$$

and so using that $d(Y_{(m)})^2/dt = 2Y_{(m)} dY_{(m)}/dt$, dividing both sides by $Y_{(m)}$ and multiplying by the integrating factor $e^{-C(M)(1+||D_t\sigma||_{L^\infty(\Omega)})t}$, we get:

$$\sup_{0\leq t\leq T}\big(||D_t \varphi^m(t)||_{L^2} + ||\sqrt{\sigma}\widetilde{\partial}\varphi^m(t)||_{L^2}\big)$$

$$\leq C\Big(||D_t \varphi^m(0)||_{L^2} + ||\sqrt{\sigma}\widetilde{\partial}\varphi^m(0)||_{L^2} + \int_0^T ||\mathcal{F}(\tau)||_{L^2} d\tau\Big). \tag{2.F.25}$$

where $C = C(M, \sup_{0\leq t\leq T} ||\widetilde{x}||_r + ||\sigma(t)||_{L^\infty}, T)$. Using the orthogonality of the e_k, we have

$$||D_t \varphi^m(0)||_{L^2(\Omega)} + ||\sqrt{\sigma}\partial\varphi^m(0)||_{L^2(\Omega)} \leq ||\varphi_1||_{L^2(\Omega)} + ||\sqrt{\sigma}\varphi_0||_{H^1(\Omega)},$$

which proves the first part of (2.F.21). We now control $||D_t^2 \varphi^m||_{H^{-1}(\Omega)}$.

Let $v \in H_0^1(\Omega)$ so that $||v||_{H^1(\Omega)} = 1$, and split $v = v^1 + v^2$ with v^1 in the span of $e_1, ..., e_m$.

Then we have:

$$\langle D_t^2\varphi^m, v\rangle = \langle D_t^2\varphi^m, v^1\rangle = (D_t^2\varphi^m, v^1)_{L^2} = -(\sigma\widetilde{\partial}\varphi^m, \widetilde{\partial}v^1)_{L^2} + (\mathcal{F}, v^1)_{L^2}. \tag{2.F.26}$$

The right-hand side is bounded by $C(M)(||D_t\varphi^m||_{L^2} + c_0||\widetilde{\partial}\varphi^m||_{L^2(\Omega)} + ||\mathcal{F}||_{L^2})||v^1||_{H^1(\Omega)}$. Noting that $||v^1||_{H^1(\Omega)} \leq ||v||_{H^1(\Omega)} = 1$ and integrating in time gives the bound for $||D_t^2\varphi^m||_{L^2(0,T;H^{-1}(\Omega))}$.

□

Lemma 2.F.2. *With assumptions as in Proposition 2.F.1, there is a unique $\varphi \in C([0,T]; H_0^1(\Omega))$ satisfying (2.F.1)-(2.F.2) with*

$$D_t\varphi \in L^\infty(0,T;L^2(\Omega)), \qquad D_t^2\varphi \in L^\infty(0,T;H^{-1}(\Omega)). \tag{2.F.27}$$

Proof. By the uniform estimate (2.F.21) and Alaoglu's theorem, passing to a subsequence we see that there is a $\varphi \in L^2(0,T;H_0^1(\Omega))$ with $D_t\varphi \in L^2(0,T;L^2(\Omega))$, $D_t^2\varphi \in L^2(0,T;H^{-1}(\Omega))$ so that $\widetilde{\partial}\varphi^m \to \widetilde{\partial}\varphi$ weakly in $L^2(0,T;L^2(\Omega))$, $D_t\varphi^m \to D_t\varphi$ weakly in $L^2(0,T;L^2(\Omega))$ and $D_t^2\varphi^m \to \varphi$ weakly in $L^2(0,T;H^{-1}(\Omega))$. Concretely, this means that if $v \in H_0^1(\Omega)$ then:

$$\int_0^T\!\!\int_\Omega (\widetilde{\partial}^k\varphi^m(t,y))\widetilde{\partial}^k v(t,y)\,dydt \to \int_0^T\!\!\int_\Omega (\widetilde{\partial}^k\varphi(t,y))\widetilde{\partial}^k v(t,y)\,dydt, \qquad k=0,1, \tag{2.F.28}$$

and

$$\langle D_t^2\varphi^m, v\rangle \to \langle D_t^2\varphi, v\rangle. \tag{2.F.29}$$

Now, given $v \in C^1([0,T]; H_0^1(\Omega))$ of the form:

$$v(t) = \textstyle\sum_{k=1}^M v_k(t)e_k, \tag{2.F.30}$$

we multiply the weak formulation (2.F.20) by $v_k(t)$, sum over k and integrate over $[0,T]$ to get that:

$$\int_0^T \langle D_t^2\varphi^m, v\rangle\,dt + \int_0^T\int_\Omega \sigma\delta^{ij}(\widetilde{\partial}_i\varphi^m)(\widetilde{\partial}_j v)\,dydt = \int_0^T\int_\Omega \mathcal{F}v\,dydt. \tag{2.F.31}$$

Taking $m \to \infty$ and using the above limits, we get that for v of the above form:

$$\int_0^T \langle D_t^2 \varphi, v \rangle \, dt + \int_0^T \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi)(\widetilde{\partial}_j v) \, dy dt = \int_0^T \int_\Omega \mathcal{F} v \, dy dt. \tag{2.F.32}$$

Since such v are dense in $L^2(0,T;H_0^1(\Omega))$ this holds for any v in this space. Hence for almost every t

$$\langle D_t^2 \varphi(t), v(t) \rangle + \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi(t))(\widetilde{\partial}_j v(t)) \, dy = \int_\Omega \mathcal{F}(t) v(t) dy, \qquad v \in H_0^1(\Omega). \tag{2.F.33}$$

By an approximation argument and the fundamental theorem of calculus, using that φ and its time derivatives are all in L^2 in time, we also get that $\varphi \in C([0,T];L^2(\Omega))$ and $D_t \varphi \in C([0,T];H^{-1}(\Omega))$ (see [12]). Hence (2.F.2) makes sense. We now have to check that $\varphi(0) = \varphi_0$ and $D_t\varphi(0) = \varphi_1$. Let $v \in C^2([0,T];H_0^1(\Omega))$ be such that $v(T) = D_t v(T) = 0$ and integrate by parts twice in time in (2.F.32) to get:

$$\begin{aligned}\int_0^T \int_\Omega D_t^2 v(t) \varphi(t) \, dy dt + \int_0^T \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi(t))(\widetilde{\partial}_j v(t)) \, dy dt \\ = \int_0^T \int_\Omega \mathcal{F}(t) v(t) \, dy dt + \int_\Omega v(0) D_t \varphi(0) - D_t v(0) \varphi(0) \, dy.\end{aligned} \tag{2.F.34}$$

On the other hand we can integrate by parts twice in (2.F.31) and take $m \to \infty$ to get also:

$$\begin{aligned}\int_0^T \int_\Omega D_t^2 v(t) \varphi(t) \, dy dt + \int_0^T \int_\Omega \sigma \delta^{ij} (\widetilde{\partial}_i \varphi(t))(\widetilde{\partial}_j v(t)) \, dy dt \\ = \int_0^T \int_\Omega \mathcal{F}(t) v(t) \, dy dt + \int_\Omega v(0) \varphi_1 - D_t v(0) \varphi_0 \, dy.\end{aligned} \tag{2.F.35}$$

Comparing these expressions using that $v(0)$ and $D_t v(0)$ are arbitrary, we have $\varphi(0) = \varphi_0$ and $D_t \varphi = \varphi_1$. □

We now want to show that we get improved regularity of φ when φ_0, φ_1 and $\mathcal{F}$ are more regular. The first step is to show that the coefficients d_m^ℓ are more regular in this case and for this we take time derivatives of the equation (2.F.1). We apply $n \le r-1$ time derivatives to (2.F.1) and write $D_t\widetilde{\Delta}\varphi = \delta^{ij}\widetilde{\partial}_j(D_t\widetilde{\partial}_i\varphi) - \delta^{ij}(\widetilde{\partial}_j V^\ell)\widetilde{\partial}_\ell\widetilde{\partial}_i\varphi$. We write the result as:

$$D_t^{n+2}\varphi - \delta^{ij}\widetilde{\partial}_j(\sigma D_t^n\widetilde{\partial}_i\varphi) - \delta^{ij}\widetilde{\partial}_j(D_t\sigma D_t^{n-1}\widetilde{\partial}_i\varphi) + \delta^{ij}\widetilde{\partial}_{j'}\big((\widetilde{\partial}_j S_\varepsilon V^{j'})D_t^{n-1}\widetilde{\partial}_i\varphi\big) = F^n, \quad (2.F.36)$$

where:

$$F^n = D_t^n F - \textstyle\sum_{s=2}^n (D_t^s\sigma)(D_t^{n-s}\widetilde{\Delta}\varphi)$$

$$+ \textstyle\sum_{s=1}^n \delta^{ij}(D_t^s A^b_j)(D_t^{n-s}\partial_b\widetilde{\partial}_i\varphi) + \delta^{ij}(\widetilde{\partial}_j\sigma + \widetilde{\partial}_j D_t\sigma)D_t^n\widetilde{\partial}_i\varphi - \delta^{ij}(\widetilde{\partial}_{j'}\widetilde{\partial}_j S_\varepsilon V^{j'})D_t^{n-1}\widetilde{\partial}_i\varphi. \quad (2.F.37)$$

We write (2.F.36) like this because the third and fourth terms have as many space derivatives of φ as the second term but fewer time derivatives, and so we will need to integrate by parts in space and time to handle them. The terms in F^k will be lower-order and can be bounded in L^2 directly.

Multiplying this by arbitrary $v \in H^1_0(\Omega)$ and integrating by parts leads to the equation:

$$\int_\Omega (D_t^{n+2}\varphi)v\,dy + \int_\Omega \sigma\delta^{ij}(D_t^n\widetilde{\partial}_i\varphi)(\widetilde{\partial}_j v)\,dy$$

$$+ \int_\Omega \delta^{ij}(D_t^{n-1}\widetilde{\partial}_i\varphi)\big(D_t\sigma(\widetilde{\partial}_j v)\widetilde{\partial}_j S_\varepsilon V^{j'})(\widetilde{\partial}_{j'}v)\big)\,dy = \int_\Omega F^n v\,dy. \quad (2.F.38)$$

With d_m^ℓ defined by (2.F.16), suppose that $d_m^\ell \in C^n([0,T])$ for some $n \ge 1$ and define:

$$B_k^n = B_k^n(d_m, ..., D_t^n d_m) = \textstyle\sum_{\ell\le k}\int_\Omega \sigma\delta^{ij}D_t^n(d_m^\ell\widetilde{\partial}_i e_\ell)\widetilde{\partial}_j e_k\,dy, \quad (2.F.39)$$

$$C_k^n = C_k^n(d_m, ..., D_t^{n-1}d_m) = \textstyle\sum_{\ell\le k}\int_\Omega \delta^{ij}D_t^{n-1}(d_m^\ell\widetilde{\partial}_i e_\ell)\big(D_t\sigma\widetilde{\partial}_j e_k + (\widetilde{\partial}_j S_\varepsilon V^{j'})\widetilde{\partial}_{j'}e_k\big)\,dy. \quad (2.F.40)$$

Also let $F^n(d_m)$ be F^n with φ replaced by $\varphi^m = \sum_{k\le m} d_m^k e_k$. Let $\dot{d}_m^1, ..., \dot{d}_m^k$ solve the ODE:

$$D_t \dot{d}_m^k + B_k^n + C_k^n = \int_\Omega F^n(d_m) e_k \, dy, \quad \dot{d}_m^k(0) = (\varphi_n, e_k)_{L^2(\Omega)}, \quad k = 1, ..., m, \tag{2.F.41}$$

where φ_n is defined by (2.F.4). By the existence and uniqueness theorem for ODE, it follows that $\dot{d}_m^k(t) = D_t^n d_m^k(t)$ for $0 \le t \le T$ and this implies that $d_m^k \in C^{n+1}(0,T)$.

Before proving that the sequence φ^m converges in stronger topologies, we will need to ensure that φ satisfies the equation (2.F.1) almost everywhere. We start with:

Lemma 2.F.3. *Suppose that the hypotheses of Proposition 2.F.1 hold. Let φ be as in Lemma 2.F.2. If $\varphi_0 \in H^2(\Omega)$, $\varphi_1 \in H_0^1(\Omega)$ and $D_t\mathcal{F} \in L^2(0,T;L^2(\Omega))$, then we have the improved regularity:*

$$D_t\varphi \in C([0,T]; H_0^1(\Omega)), \quad D_t^2\varphi \in L^\infty([0,T]; L^2(\Omega)), \quad D_t^3\varphi \in L^\infty([0,T]; H^{-1}(\Omega)), \tag{2.F.42}$$

Proof. Take $n = 1$, multiply (2.F.41) by $D_t^2 d_m^k$, use $\dot{d}_m^k = D_t d_m^k$ and write:

$$\widetilde{\partial}_j e_k D_t^2 d_m^k = \widetilde{\partial}_j D_t^2 \varphi^m = D_t^2 \widetilde{\partial}_j \varphi^m - (D_t^2 A^a{}_j)\partial_a \varphi^m - 2(D_t A^a{}_j) D_t \partial_a \varphi^m \equiv D_t^2 \widetilde{\partial} \varphi^m + R_j^1, \tag{2.F.43}$$

which gives:

$$B_k^1 D_t^2 d_m^k = \int_\Omega \sigma \delta^{ij} (D_t \widetilde{\partial}_i \varphi^m)(\widetilde{\partial}_j e_k D_t^2 d_m^k) \, dy = \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \sigma\delta^{ij}(D_t\widetilde{\partial}_i\varphi^m)(D_t\widetilde{\partial}_j\varphi^m)\,dy\Big)$$
$$- \int_\Omega (D_t\sigma)\delta^{ij}(D_t\widetilde{\partial}_i\varphi^m)(D_t\widetilde{\partial}_j\varphi^m)\,dy - \int_\Omega \sigma\delta^{ij} D_t\widetilde{\partial}_i\varphi^m\big((D_t^2A^a{}_j)\partial_a\varphi^m - 2(D_tA^a{}_j)D_t\partial_a\varphi^m\big)\,dy. \tag{2.F.44}$$

and similarly:

$$C_k^1 D_t^2 d_m^k = \frac{dC_1}{dt} - \int_\Omega (\widetilde{\partial}_i\varphi^m)\big((D_t^2\sigma)(D_t\widetilde{\partial}_j\varphi^m) - D_t\big(\sigma(\widetilde{\partial}_j S_\varepsilon V^{j'})\big)(D_t\widetilde{\partial}_{j'}\varphi^m)\big)\,dy$$
$$- \int_\Omega \sigma\delta^{ij}(D_t\widetilde{\partial}_i\varphi^m)\big(R_j^1 - (\widetilde{\partial}_j S_\varepsilon V^{j'})R_{j'}^1\big)\,dy, \tag{2.F.45}$$

where:

$$C_1 = C_1[\varphi^m] = \int_\Omega (D_t\sigma)\delta^{ij}(\widetilde{\partial}_i\varphi^m)(D_t\widetilde{\partial}_j\varphi^m) - \delta^{ij}\sigma(\widetilde{\partial}_j S_\varepsilon V^{j'})(\widetilde{\partial}_i\varphi^m)(D_t\widetilde{\partial}_{j'}\varphi^m)dy. \qquad (2.F.46)$$

By Sobolev embedding and (2.D.2):

$$||D_t^2 A^a{}_j||_{L^\infty(\Omega)} + ||D_t A^a{}_j||_{L^\infty(\Omega)} + ||D_t\widetilde{\partial}_j S_\varepsilon V^\ell||_{L^\infty(\Omega)} \leq C(M)||\widetilde{x}||_r, \qquad (2.F.47)$$

$$||D_t\sigma||_{L^\infty(\Omega)} + ||\widetilde{\partial} D_t\sigma|||_{L^\infty(\Omega)} + ||D_t^2\sigma||_{L^\infty(\Omega)} \leq C||\sigma||_r. \qquad (2.F.48)$$

With $Y_m^1 = ||D_t^2\varphi^m||_{L^2(\Omega)} + ||\sqrt{\sigma}D_t\widetilde{\partial}\varphi^m||_{L^2(\Omega)}$, the above calculation shows that:

$$\frac{d}{dt}\Big((Y_m^1)^2 - C_1[\varphi^m]\Big) \leq C(M, ||\widetilde{x}||_r)(1+||\sigma||_r)\Big(Y_m^1 + Y_m + ||F_m^1||_{L^2(\Omega)}\Big)Y_m^1. \qquad (2.F.49)$$

Multiplying both sides by the integrating factor $e^{-C(M,||\widetilde{x}||_r)(1+||\sigma||_r)t}$, integrating, and then using that $C[\varphi^m] \leq C(M)||\widetilde{x}||_r(\delta(Y_m^1)^2 + \delta^{-1}Y_m^2)$ for any $\delta > 0$, this implies that:

$$Y_m^1(t)^2 \leq C(M, ||\widetilde{x}||_r, ||\sigma||_r)\Big(Y_m^1(0)^2 + Y_m(t)^2 + \int_0^t Y_m^1(\tau)^2 + Y_m(\tau)^2 + ||D_tF_m^1(\tau)||^2_{L^2(\Omega)}\,d\tau\Big), \qquad (2.F.50)$$

and so by Grönwall's integral inequality, this implies:

$$\sup_{0\leq t\leq T} Y_m^1(t) \leq C\Big(Y_m^1(0) + \sup_{0\leq t\leq T} Y_m(t) + \int_0^T (1+||\sigma||_r)Y_m(\tau) + ||F_m^1(\tau)||_{L^2(\Omega)}\,d\tau\Big). \qquad (2.F.51)$$

Arguing as in the previous lemma, this implies that the sequence $D_t\varphi^m$ has limit $\dot{\varphi}$ with:

$$D_t\dot{\varphi} \in L^\infty(0,T;L^2(\Omega)), \qquad D_t^2\dot{\varphi} \in L^\infty(0,T;H^{-1}(\Omega)). \qquad (2.F.52)$$

Since also $\varphi^m \to \dot{\varphi}$ in L^2 by the previous lemma, it follows that $\varphi = \dot{\varphi}$ and in particular we get the first two statements in (2.F.42). To get that $D_t^3\varphi \in L^\infty([0,T];H^{-1}(\Omega))$, we argue as in the previous lemma. Also, since the compatibility conditions hold, we have that $Y_m^1(0) \to Y^1(0) = ||\varphi_2||_{L^2(\Omega)} + ||\sqrt{\sigma(0)}\widetilde{\partial}\varphi_1||_{L^2(\Omega)}$. □

We can now prove that φ has enough regularity that the elliptic estimates from the Section 2.B hold:

Lemma 2.F.4. *If $\varphi_0 \in H^2(\Omega), \varphi_1 \in H^1_0(\Omega)$ and (2.5.2),(2.F.11) hold, there is a constant $C_1 = C_1(M, K, T)$ so that:*

$$\text{ess sup}_{0\le t\le T}\big(||\varphi(t)||_{H^2(\Omega)} + ||D_t\varphi(t)||_{H^1_0(\Omega)} + ||D_t^2\varphi(t)||_{L^2(\Omega)}\big) + ||D_t^3\varphi||_{L^2(0,T;H^{-1}(\Omega))}$$
$$\le C_1\big(||\mathcal{F}||_{H^1(0,T;L^2(\Omega)} + ||\varphi_0||_{H^2(\Omega)} + ||\varphi_1||_{H^1(\Omega)}\big). \quad (2.F.53)$$

Proof. By the previous lemma, we already have the second, third and fourth estimates in (2.F.53) and it just remains to bound the first term. The point is that we do not yet know that the wave equation (2.F.1) holds almost everywhere so we cannot use the elliptic estimate (2.5.8). As in [12], will instead prove an elliptic estimate for the approximate solution φ^m. We let $\{\lambda_\ell\}_{\ell=0}^\infty$ be the eigenvalues of Δ on $H^1_0(\Omega)$. Multiplying both sides of (2.F.16) by $\lambda_\ell d^\ell_{(m)}$ and summing from $\ell = 1$ to m, we get that:

$$\int_\Omega \sigma\delta^{ij}(\widetilde{\partial}_i\varphi^m)(\widetilde{\partial}_j\Delta\varphi^m)\,dy = \int_\Omega(\mathcal{F} - D_t^2\varphi^m)\Delta\varphi^m\,dy. \quad (2.F.54)$$

Since $\Delta\varphi^m = 0$ on $\partial\Omega$, we integrate by parts in the left-hand side and use the estimate (2.B.31), which gives:

$$||\widetilde{\partial}\varphi^m||_{H^1(\Omega)} \le C(M)\big(||D_t^2\varphi^m||_{L^2(\Omega)} + ||D_t\widetilde{\partial}\varphi^m||_{L^2(\Omega)} + ||\widetilde{\partial}\varphi^m||_{L^2(\Omega)} + ||\varphi^m||_{L^2(\Omega)}\big).$$
$$(2.F.55)$$

Since $\varphi \in H^2(\Omega)$, we now have that φ solves the equation (2.F.1)- (2.F.2) a.e. in $[0,T]\times\Omega$. □

Proof of Proposition 2.F.1. We argue by induction. We have just shown that the theorem holds for $s = 0, 1$. We suppose that the theorem holds for $s = 1, ..., n-1 \le r-1$ and we now assume that the compatibility conditions (2.F.10) hold for $s = 0, ..., n$. By the inductive assumption, there is a unique φ satisfying the equation (2.F.1)-(2.F.2) in the weak sense so that:

$$D_t^n \varphi \in L^\infty([0,T]; L^2(\Omega)), \qquad D_t^{n-\ell}\widetilde{\partial}\varphi \in L^\infty([0,T]; H^{\ell-1}(\Omega)), \qquad \ell = 0, ..., n. \tag{2.F.56}$$

Moreover, with φ^m as defined above, we have that $D_t^n \varphi^m(t) \to D_t^s \varphi(t)$ in $L^2(\Omega)$ and $D_t^{n-\ell}\widetilde{\partial}\varphi^m(t) \to D_t^{n-\ell}\widetilde{\partial}\varphi(t)$ in $H^{\ell-1}(\Omega)$ for $\ell = 0, ..., n$ and $0 \le t \le T$. We multiply (2.F.16) by $D_t^{n+1} d_m^\ell$ and write:

$$(\widetilde{\partial}_j e_k) D_t^{n+1} d_m^k = D_t^{n+1}\widetilde{\partial}_j \varphi^m - \textstyle\sum_{s=1}^{n+1} (D_t^s A^a{}_j) D_t^{n+1-s}\partial_a \varphi^m \equiv D_t^{n+1}\widetilde{\partial}_j\varphi^m - R_j^n, \tag{2.F.57}$$

and this leads to:

$$B_k^n D_t^{n+1} d_m^k = \frac{1}{2}\frac{d}{dt}\int_\Omega \sigma\delta^{ij}(D_t^n\widetilde{\partial}_i\varphi^m) D_t^n\widetilde{\partial}_j\varphi^m \, dy - \int_\Omega (D_t\sigma)\delta^{ij}(D_t^n\widetilde{\partial}_i\varphi^m) D_t^n\widetilde{\partial}_j\varphi^m \, dy$$
$$- \int_\Omega \sigma\delta^{ij}(D_t^n\widetilde{\partial}_i\varphi^m) R_j^n \, dy, \tag{2.F.58}$$

$$C_k^n D_t^{n+1} d_m^k = \frac{dC_n}{dt} - \int_\Omega (D_t^{n-1}\widetilde{\partial}_i\varphi^m)\big((D_t^2\sigma)(D_t^n\widetilde{\partial}_j\varphi^m) - D_t\big(\sigma(\widetilde{\partial}_j S_\varepsilon V^\ell)\big)(D_t^n\widetilde{\partial}_\ell\varphi^m)\big) \, dy$$
$$- \int_\Omega \sigma\delta^{ij}(D_t^n\widetilde{\partial}_i\varphi^m)\big(R_j^n - (\widetilde{\partial}_j S_\varepsilon V^{j'}) R_{j'}^n\big) \, dy, \tag{2.F.59}$$

where

$$C_n = \int_\Omega (D_t\sigma)\delta^{ij}(D_t^{n-1}\widetilde{\partial}_i\varphi^m)(D_t^n\widetilde{\partial}_j\varphi^m) - \delta^{ij}\sigma(\widetilde{\partial}_j S_\varepsilon V^\ell)(D_t^{n-1}\widetilde{\partial}_i\varphi^m)(D_t^n\widetilde{\partial}_\ell\varphi^m) \, dy. \tag{2.F.60}$$

Using Lemma 2.F.5 to control R^n, F^n and arguing as in the proof of Lemma 2.F.3, we get:

$$\frac{d}{dt}\Big(Y_m^n(t) - C_n[\varphi^m]\Big) \le C(M, ||\widetilde{x}||_r, ||D_t\widetilde{x}||_r, ||D_t^2\widetilde{x}||_r, ||\sigma||_r)\Big(Y_m^n + Y_m^{n-1} + ||F_m^n||_{L^2(\Omega)}\Big), \tag{2.F.61}$$

and so applying the inductive assumption (2.F.56) and arguing as in the previous lemma, we get the result. $\square$

Lemma 2.F.5. *Fix $r \geq 7$. Let F_m^n be F^n (defined in (2.F.37)) with φ replaced by φ^m and R^n be as in (2.F.57). There are constants C_r depending on $M, ||\widetilde{x}||_r, ||D_t\widetilde{x}||_r, ||D_t^2\widetilde{x}||_r, ||\sigma||_r$, so that if $k \leq r$, then:*

$$||F_m^n||_{L^2(\Omega)} + ||R^n||_{L^2(\Omega)} \leq C_r\big(||\varphi^m||_n + ||\widetilde{\partial}\varphi^m||_n + ||D_t^n F||_{L^2(\Omega)}\big). \tag{2.F.62}$$

We remark that unlike the estimates in Section 2.6, these estimates depend on $||D_t^2\widetilde{x}||_r$. This is because the estimates in that section are all in terms of $\widetilde{\partial}\varphi$ i.e. we estimate $||D_t^k\widetilde{\partial}\varphi||_{L^2(\Omega)}$, but in the above proof we are forced to consider what amounts to $||\widetilde{\partial}D_t^k\varphi^m||_{L^2(\Omega)}$. The error term this generates can be dealt with since in the application we have in mind, $D_t^2\widetilde{x} = D_t S_\varepsilon V$ behaves like $\widetilde{\partial}\varphi$.

Proof. First, we control the first two terms in F^n with φ replaced by φ^m. When $s \leq r-2$, we have:

$$||D_t^s\sigma||_{L^\infty(\Omega)}||D_t^{n-s}\widetilde{\Delta}\varphi^m||_{L^2(\Omega)} \leq ||\sigma||_r||D_t^{n-s}\widetilde{\Delta}\varphi^m||_{L^2(\Omega)}. \tag{2.F.63}$$

To control this second term, we use the commutator estimate (2.D.23):

$$||D_t^{n-s}\widetilde{\Delta}\varphi^m||_{L^2(\Omega)} \leq C(M, ||\widetilde{x}||_r)||D_t^{n-s}\widetilde{\partial}\varphi^m||_{H^1(\Omega)}. \tag{2.F.64}$$

Since $s \geq 2$, we have $||D_t^{n-s}\widetilde{\partial}\varphi^m||_{H^1(\Omega)} \leq ||\widetilde{\partial}\varphi^m||_{n-1}$. If instead $s = r-1, r$, the result is bounded by:

$$||D_t^s\sigma||_{L^2(\Omega)}||D_t^{n-s}\widetilde{\Delta}\varphi^m||_{L^\infty(\Omega)} \leq C||\sigma||_r||D_t^{n-s}\widetilde{\Delta}\varphi^m||_{H^2(\Omega)}, \tag{2.F.65}$$

and so again applying the commutator estimate, this term is bounded by the right-hand side of (2.F.62) provided $||\widetilde{\partial}\varphi^m||_{n-s+3} \leq ||\widetilde{\partial}\varphi^m||_n$, and this follows since $r \geq n \geq s$, $s = r-1, r$ and $r \geq 7$.

We now control the remaining terms from the definition of F^n. The last two terms are clearly bounded by the right-hand side of (2.F.62) so we just bound the terms in the sum. When $s \leq r-3$, we bound the terms by:

$$||D_t^s A^b_j||_{L^\infty}||D_t^{n-s}\partial_b\widetilde{\partial}_i\varphi^m||_{L^2(\Omega)} \leq ||A^b_j||_{s+2}||D_t^{n-s}\widetilde{\partial}\varphi^m||_{H^1(\Omega)} \leq C(M)||\widetilde{x}||_r||\widetilde{\partial}\varphi^m||_{n-s+1}, \tag{2.F.66}$$

and since $s \geq 2$, we have $n-s+1 \leq n-1$ as required.

We now consider the remaining cases $r-2 \leq s \leq r$. In these cases we instead bound the summands by:

$$||D_t^s A^b_j||_{L^2(\Omega)}||D_t^{n-s}\partial_b\widetilde{\partial}_i\varphi^m||_{L^\infty(\Omega)} \leq C(M, ||\widetilde{x}||_r)||D_t\widetilde{x}||_r||D_t^{n-s}\widetilde{\partial}\varphi^m||_{H^3(\Omega)}, \tag{2.F.67}$$

and since in this case $n-s+3 \leq n-1$ (because $r-2 \leq s \leq n$ and $r \geq 7$), this second factor is bounded by the right-hand side of (2.F.62) as well, and this completes the proof of the bounds for F^n_m.

We now control R^n. This follows in the same way as the bounds we have just proved but note that we also need to consider the case $s = r+1$. This is the reason that $||D_t^2\widetilde{x}||_r$ enters into the estimates. When $s \leq r-3$ we argue as above and the result is that:

$$\begin{aligned} ||D_t^s A^b_j||_{L^\infty(\Omega)}||D_t^{n+1-s}\partial\varphi^m||_{L^2(\Omega)} &\leq C||A^b_j||_{s+2}||\partial_b\varphi^m||_{n+1-s} \\ &\leq C(M, ||\widetilde{x}||_r)||\partial_b\varphi^m||_{n+1-s}. \end{aligned} \tag{2.F.68}$$

The remaining cases are $s = r-2, r-1, r, r+1$ and for these we bound the result by:

$$||D_t^s A^b_j||_{L^2(\Omega)}||D_t^{n+1-s}\partial_b\varphi^m||_{L^\infty(\Omega)} \leq C(M)||D_t^2\widetilde{x}||_r||\partial_y\varphi^m||_n, \tag{2.F.69}$$

where in the last step we used that $n+1-s \leq n$ when $s \geq r-2$ for $r \geq 7$. We now need to re-write $\partial_b \varphi^m = A^j{}_b \widetilde{\partial}_j \varphi^m$ and we note that by similar arguments to the above we have:

$$||A^j{}_b \widetilde{\partial}_j \varphi^m||_n \leq C(M, ||\widetilde{x}||_r) ||D_t \widetilde{x}||_r ||\widetilde{\partial} \varphi^m||_n. \qquad \square$$

2.F.2 Existence for a nonlinear wave equation

We assume that (2.2.18) hold and that $e : (0, \infty) \to \mathbb{R}$ is a function satisfying (2.2.18). In this section we prove that the nonlinear wave equation:

$$e'(\varphi) D_t^2 \varphi - \widetilde{\Delta} \varphi = \mathcal{F} \text{ in } [0, T] \times \Omega, \tag{2.F.70}$$

$$\varphi = 0 \text{ on } [0, T] \times \partial\Omega, \tag{2.F.71}$$

$$\varphi(0, y) = \varphi_0(y), \qquad D_t \varphi(0, y) = \varphi_1(y) \text{ on } \Omega, \tag{2.F.72}$$

has a unique strong solution φ satisfying (2.F.12). We will construct a solution so that for some $L = L[\phi] < \infty$:

$$\sum_{k+|J| \leq 3} |D_t^k \partial_y^J \widetilde{\partial} \varphi| + |D_t^k \varphi| \leq L, \quad \text{ in } [0, T] \times \Omega. \tag{2.F.73}$$

We assume that $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ where $\mathcal{F}_1 = \mathcal{F}_1(t, y)$ is a function and $\mathcal{F}_2 = \mathcal{F}_2[\varphi]$ is a functional so that there are continuous functions $N_s[\phi] = N_s(L[\phi], ||\varphi||_{s-1}, ||\varphi||_{s,0})$, $N_s'[\phi] = N_s'(L[\phi], ||\varphi||_{s-1})$ so that:

$$||D_t^s \mathcal{F}_2[\varphi]||_{L^2(\Omega)} \leq N_s[\varphi](||D_t^{s+1} \varphi||_{L^2(\Omega)} + ||\varphi||_{L^2(\Omega)}), \qquad ||\mathcal{F}_2[\varphi]||_{s-1} \leq N_s'[\varphi] ||\varphi||_s. \tag{2.F.74}$$

We will additionally assume that if f, g satisfy (2.F.73) there are continuous functions $N_s''[f, g]$, $N_s'''[f, g]$ depending on $L[f], L[g], ||f||_s, ||g||_s$ with N_s'' depending additionally on $||f||_{s+1,0}, ||g||_{s+1,0}$ so that:

$$||D_t^s\mathcal{F}_2[f]-D_t^s\mathcal{F}_2[g]||_{L^2(\Omega)}\leq N_s''[f,g]||f-g||_{s+1,0}, \tag{2.F.75}$$

$$||\mathcal{F}_2[f]-\mathcal{F}_2[g]||_{s-1}\leq N_s'''[f,g]||f-g||_s. \tag{2.F.76}$$

In Section 2.8 we take $F_2=e''(\varphi)(D_t\varphi)^2+\rho[\varphi]$, which satisfies these estimates. The energies we use are:

$$Y_s(t)=\Big(\frac{1}{2}\sum\nolimits_{k\leq s}\int_\Omega e'(\varphi)|D_t^{k+1}\varphi|^2+\delta^{ij}(D_t^k\widetilde{\partial}_i\varphi)(D_t^k\widetilde{\partial}_j\varphi)\,\widetilde{\kappa}dy\Big)^{1/2}. \tag{2.F.77}$$

We recursively define:

$$\varphi_k=\big(e'(\varphi)^{-1}(D_t^{k-2}\widetilde{\Delta}\varphi+\mathcal{F}_1+\mathcal{F}_2[\varphi])\big)\big|_{t=0}, \tag{2.F.78}$$

and we say that φ_0,φ_1 satisfy the compatibility conditions to order s if:

$$\varphi_k\in H_0^1(\Omega),\qquad k=0,...,s. \tag{2.F.79}$$

Theorem 2.F.1. *Fix $r\geq 7$ and suppose that $V\in\mathcal{X}^{r+1}(T_1)$ for some $T_1>0$ satisfies (2.9.12) and that the bound (2.5.2) holds. Take K so that*

$$\sup\nolimits_{0\leq t\leq T_1}\big(||\widetilde{x}(t)||_r+||V(t)||_r+||D_tV(t)||_r+||D_t\mathcal{F}_1(t)||_{r-1}+||\mathcal{F}_1(t)||_{r-1}\big)\leq K. \tag{2.F.80}$$

Suppose that (2.F.6)-(2.F.9), (2.2.18) and the compatibility conditions (2.F.79) hold for some $s\leq r$. Let L_0 satisfy:

$$\sum\nolimits_{k+|J|\leq 3}||\partial_y^L\widetilde{\partial}\varphi_k^\varepsilon||_{L^\infty(\Omega)}+||\varphi_k^\varepsilon||_{L^\infty(\Omega)}\leq L_0. \tag{2.F.81}$$

There is a continuous function G_r' so that if T satisfies:

$$TG_r'(M,L_0,L_0^{-1},Y_r(0),K,T_1)\leq 1,\quad \textit{and}\quad T\leq T_1, \tag{2.F.82}$$

the problem (2.F.70)-(2.F.72) *has a unique solution* φ *satisfying:*

$$D_t^s\varphi \in L^\infty([0,T];L^2(\Omega)), \qquad D_t^{s+1-\ell}\widetilde{\partial}\varphi \in L^\infty([0,T];H^{\ell-1}(\Omega)), \qquad \ell = 0,...,s+1, \tag{2.F.83}$$

and there are constants $\mathcal{C}_s$ *depending on* $M, L_0,\ Y_s(0), K,$ *and* T *so that the following estimates hold:*

$$Y_s(t) \leq \mathcal{C}_s\Big(Y_s(0) + \int_0^t ||\mathcal{F}_1(\tau)||_{s,0} + ||\mathcal{F}_1(\tau)||_{s-1}\, d\tau\Big), \tag{2.F.84}$$

$$||\widetilde{\partial}\varphi(t)||_s \leq \mathcal{C}_s\big(Y_s(t) + ||F_1(t)||_{s-1}\big), \tag{2.F.85}$$

for $0 \leq t \leq T$ *and*

$$\textstyle\sum_{k+\ell\leq 2}|\partial^\ell D_t^k\varphi(t,y)| \leq 2L_0, \qquad in\ [0,T]\times\Omega. \tag{2.F.86}$$

We will construct solutions to (2.F.70)-(2.F.72) by considering the sequence φ^ν, $\nu = 0,1,...,$ defined by:

$$\varphi^0 = \textstyle\sum_{k=0}^s \varphi_k^\varepsilon t^k/k!, \tag{2.F.87}$$

$$D_t^2\varphi^\nu - e'(\varphi^{\nu-1})^{-1}\widetilde{\Delta}\varphi^\nu = e'(\varphi^{\nu-1})^{-1}\mathcal{F}^{\nu-1}, \text{ in } [0,T]\times\Omega, \tag{2.F.88}$$

$$\varphi^\nu = 0 \text{ on } [0,T]\times\partial\Omega, \tag{2.F.89}$$

$$\varphi^\nu(0,y) = \varphi_0(0,y), \qquad D_t\varphi^\nu(0,y) = \varphi_1(0,y), \text{ on } \Omega, \tag{2.F.90}$$

with $\mathcal{F}^{\nu-1} = \mathcal{F}_1 + \mathcal{F}_2[\varphi^{\nu-1}]$ and where φ_k^ε are defined in (2.4.15). Note that with this choice of φ^0, we have that $D_t^j\varphi^0|_{t=0} = \varphi_j^\varepsilon$, $j \leq s$. This system also has compatibility conditions which must be satisfied to construct a sufficiently regular solution. We recursively define the sequence $\varphi_k^\nu, k \geq 2$ by:

$$\varphi_k^\nu = \big(e'(\varphi^{\nu-1})^{-1}\big(D_t^{k-2}\widetilde{\Delta}\varphi^\nu + D_t^{k-2}\mathcal{F}^{\nu-1} + \mathcal{G}_k[\varphi^\nu,\varphi^{\nu-1}]\big)\big|_{t=0}, \tag{2.F.91}$$

where we are writing:

$$\mathcal{G}_k[\varphi^\nu,\varphi^{\nu-1}] = D_t^{k-2}\big(e'(\varphi^{\nu-1})D_t^2\varphi^\nu\big) - e'(\varphi^{\nu-1})D_t^k\varphi^\nu. \tag{2.F.92}$$

The compatibility conditions for the system (2.F.89)-(2.F.90) are then the requirement that:

$$\varphi_k^\nu \in H_0^1(\Omega), \quad k\geq 2. \tag{2.F.93}$$

ince $\varphi_0^\nu = \varphi_0, \varphi_1^\nu = \varphi_1$ and both of these sequences are defined recursively, from (2.F.4) and (2.F.91) it follows that $\varphi_k^\nu = \varphi_k$ for all $\nu\geq 0$ and so the compatibility conditions for the approximate problem (2.F.89)-(2.F.90) are satisfied so long as the compatibility conditions (2.F.79) for the nonlinear problem (2.F.70)-(2.F.72) hold.

We now argue by induction to show that the above problem has a unique solution with bounds that hold uniformly in ν. Let X_T^r be closure of $C^\infty([0,T];C^\infty(\Omega))$ with respect to the norm:

$$||\varphi||_{X_T^r} = \sup{}_{0\leq t\leq T}\textstyle\sum_{s=0}^r ||D_t^{s+1}\varphi(t)||_{L^2(\Omega)} + ||D_t^s\widetilde{\partial}\varphi(t)||_{H^{r-s}(\Omega)}. \tag{2.F.94}$$

Assume that for $\nu\geq 1$ we have a solution $\varphi^{\nu-1}\in X_T^r$ which moreover satisfies (2.F.86). Writing:

$$Y_s^{\nu-1}(t) = \textstyle\sum_{k\leq s}\Big(\frac{1}{2}\int_\Omega e'(\varphi^{\nu-2})|D_t^{k+1}\varphi^{\nu-1}(t)|^2 + |D_t^s\widetilde{\partial}\varphi^{\nu-1}|^2\,\widetilde{\kappa}dy\Big)^{1/2}, \tag{2.F.95}$$

by Proposition 2.F.1, we have the estimates:

$$Y_s^{\nu-1}(t) \leq \mathcal{C}_s\Big(Y_s(0) + \int_0^t ||D_t^{s-1}\mathcal{F}_1(\tau)||_{L^2(\Omega)}\,d\tau\Big), \tag{2.F.96}$$

$$||\widetilde{\partial}\varphi^{\nu-1}||_s \leq \mathcal{C}_s\big(Y_s + ||\mathcal{F}_1||_{s-1}\big), \quad s=0,...,r. \tag{2.F.97}$$

where here $\mathcal{C}_s$ depends on $M, Y_{s-1}(0)$, K and $\sup_{0\leq t\leq T}||e'(\varphi^{\nu-1}(t))||_r$. Note that we are using that $Y_s^{\nu-1}(0) = Y_s(0)$ in (2.F.97). By these estimates, (2.F.74) and Lemma 2.D.8 to control $e'(\varphi^{\nu-1})$, we have:

$$||\mathcal{F}^{\nu-1}||_{s,0} + ||\mathcal{F}^{\nu-1}||_{s-1} + ||\sigma(\varphi^{\nu-1})||_{s,0} + ||\sigma(\varphi^{\nu-1})||_r \leq C_s(M, L_0, Y_r(0), K). \tag{2.F.98}$$

By (2.F.1), there is a unique $\varphi^\nu \in X_T^r$ satisfying (2.F.89)-(2.F.90) and so that (2.F.12) holds. By the above estimates and the inductive assumption we also have:

$$Y_s^\nu(t) \leq \mathcal{C}_s\Big(Y_s(0) + \int_0^T ||\mathcal{F}_1(\tau)||_{s,0} + ||\mathcal{F}_1(\tau)||_{s-1}\, d\tau\Big), \tag{2.F.99}$$

where $\mathcal{C}_s = \mathcal{C}_s(M, L_0, Y_r(0), K)$ and we again are using that $Y_s(0)$ is independent of ν. We note that by Sobolev embedding, the estimate (2.F.99) and the estimate (2.6.8), just as in the proof of Corollary 2.6.1, we have that:

$$L^\nu(t) \equiv \textstyle\sum_{k+|J|\leq 3} |\partial_y^J D_t^k \widetilde{\partial} \varphi^\nu(t,y)| + |D_t^k \varphi^\nu(t,y)| \leq L_0 + TP_0^\nu,, \tag{2.F.100}$$

where

$$P_0^\nu \equiv P_0^\nu(M, \sup\nolimits_{0\leq t\leq T} L^\nu(t), Y_5^\nu(0), K) \tag{2.F.101}$$

and so a continuity argument (see the proof of Corollary 2.6.1) gives that $\sup_{0\leq t\leq T} L^\nu(t) \leq 2L_0$ provided that $T(2L_0)^{-1} P_0^\nu(M, 2L_0, Y_5^\nu(0), K) \leq 1$. Note that in fact P_0 is independent of ν since $Y_5^\nu(0)$ is.

The sequence φ^ν is therefore uniformly bounded in $X_{T_0}^r$ for a fixed $T_0 > 0$, and therefore there is a $\varphi \in X_{T_0}^r$ so that $\varphi^\nu \to \varphi$ weakly. We now show that there is $T^* = T^*(M, L_0, Y_r(0), K) \leq T_0$ so that if $T_1 \leq T^*$ then

$$||\varphi^\nu - \varphi^{\nu-1}||_{X_{T_1}^0} \leq 2^{-1} ||\varphi^{\nu-1} - \varphi^{\nu-2}||_{X_{T_1}^0}. \tag{2.F.102}$$

Assuming that this holds for the moment, it follows that the sequence φ^ν is a Cauchy sequence in $X_{T_1}^0$ and so converges strongly to some $\tilde{\varphi} \in X_{T_1}^0$. This limit has to coincide with the φ

above and in particular this shows that the φ^ν converges strongly to φ, and so φ satisfies the nonlinear equation (2.F.70).

To prove (2.F.102), we take $T^* \leq T_0$ and set $\psi = \varphi^\nu - \varphi^{\nu-1}$ and note that with $\mathcal{F}^{\nu,\nu-1} = \mathcal{F}_2^\nu - \mathcal{F}_2^{\nu-1}$ we have:

$$e'(\varphi^\nu) D_t^2 \psi - \widetilde{\Delta}\psi = \mathcal{F}^{\nu,\nu-1} + (e'(\varphi^\nu) - e'(\varphi^{\nu-1})) D_t^2 \varphi^{\nu-1}, \tag{2.F.103}$$

$$\psi|_{[0,T]\times\partial\Omega} = 0, \tag{2.F.104}$$

$$\psi|_{t=0} = D_t\psi|_{t=0} = 0. \tag{2.F.105}$$

By the estimates (2.F.76), the estimate (2.F.13), and the product estimate (2.A.45), we have that:

$$\begin{aligned} Y_0^{\nu,\nu-1}(t) &\equiv \Big(\frac{1}{2}\int_\Omega e'(\varphi^\nu)|D_t\psi|^2 + |\widetilde{\partial}\psi|^2\,\widetilde{\kappa} dy\Big)^{1/2} \\ &\leq \mathcal{C}_0 \int_0^T ||\varphi^{\nu-1} - \varphi^{\nu-2}||_1\, dt \leq \mathcal{C}_0 T ||\varphi^{\nu-1} - \varphi^{\nu-2}||_{X_T^0}, \end{aligned} \tag{2.F.106}$$

where $\mathcal{C}_s = \mathcal{C}_s(M, L_0, Y_r(0), K)$. Since $||\varphi^\nu - \varphi^{\nu-1}||_{X_T^0} \lesssim \sup_{0\leq t\leq T} Y_0^{\nu,\nu-1}$, taking T sufficiently small gives (2.F.102).

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Chapter 3

Gravity water waves in three space dimensions with vorticity

3.1 Introduction

The motion of an inviscid incompressible fluid occupying a region $\mathcal{D} = \cup_{0 \le t \le T}\{t\} \times \mathcal{D}_t$, $\mathcal{D}_t \subset \mathbb{R}^3$ is described by the fluid velocity $v = (v_1, v_2, v_3)$ and a non-negative function p known as the pressure. If the fluid body is subject to a uniform vertical gravitational force, then the equations of motion are given by Euler's equations:

$$\big(\partial_t + v^k \partial_k\big) v_i = -\partial_i p - e_3 \text{ in } \mathcal{D}_t, \quad \text{where } \partial_i = \frac{\partial}{\partial x^i}, e_3 = (0,0,1), \tag{3.1.1}$$

and conservation of mass:

$$\operatorname{div} v = \partial_i v^i = 0, \quad \text{in } \mathcal{D}_t. \tag{3.1.2}$$

Here, we are using the Einstein summation convention and summing over repeated upper and lower indices and writing $v^i = \delta^{ij} v_j$. We have also chosen units so that the acceleration due to gravity is one. Fluid particles on the boundary move with the velocity of the fluid:

$$v \cdot n = \kappa, \tag{3.1.3}$$

where κ is the normal velocity of $\partial\mathcal{D}_t$ and n is the unit normal to $\partial\mathcal{D}_t$. We assume that $\mathcal{D}_t$ is given by $\mathcal{D}_t = \{(x_1, x_2, y) : x_1, x_2 \in \mathbb{R}^2, y \leq h(t, x_1, x_2)\}$ for some function h, in which case (3.1.3) can be re-written as:

$$\partial_t h + v^1\partial_1 h + v^2\partial_2 h = v^3 \quad \text{on } \partial\mathcal{D}_t. \tag{3.1.4}$$

If the fluid body moves in vaccuum and there is no surface tension on the boundary then the pressure satisfies:

$$p = 0 \text{ on } \partial\mathcal{D}_t. \tag{3.1.5}$$

Given $h_0 : \mathbb{R}^2 \to \mathbb{R}$, set $\mathcal{D}_0 = \{(x_1, x_2, y) | y \leq h_0(x_1, x_2)\}$. If $v_0 : \mathcal{D}_0 \to \mathbb{R}^3$ is a vector field satisfying the constraint $\operatorname{div} v_0 = 0$, we want to find a function h and a vector field v so that with $\mathcal{D}_t = \{(x_1, x_2, y) | y \leq h(t, x_1, x_2)\}$, v satisfies (3.1.1)-(3.1.2) and the initial conditions:

$$h(0, x_1, x_2) = h_0(x_1, x_2), \quad v = v_0 \quad \text{on } \{0\} \times \mathcal{D}_0. \tag{3.1.6}$$

This problem is ill-posed unless the following "Taylor sign condition" holds (see [1]):

$$-n \cdot \partial p(x,t) \geq \delta_0 > 0 \text{ on } \partial\mathcal{D}_t, \quad \text{where } n \cdot \partial = n^i \nabla_i, \tag{3.1.7}$$

where n denotes the unit normal to $\partial\mathcal{D}_t$. This condition ensures that the pressure is positive in the interior of the fluid and prevents the Rayleigh-Taylor instability from occuring.

In the irrotational case ($\omega \equiv \operatorname{curl} v = 0$), the velocity v is given by $v = \nabla\psi$ for a harmonic function $\psi : \mathcal{D}_t \to \mathbb{R}$, and the motion of the fluid is determined entirely by h and $\varphi = \psi|_{\partial\mathcal{D}_t}$. This, and related problems have been studied extensively by several authors in the case that the fluid domain $\mathcal{D}_t$ is diffeomorphic to the half-space. See for example [2], [3], [4], as well as [5] for a recent overview of these problems. Let us single out the works [3], [4], in which the authors proved that in the irrotational case, (3.1.1)-(3.1.5) is globally well-posed for sufficiently small and well-localized initial data.

In the case that $\omega \neq 0$, Lindblad-Christodoulou [6] used the Taylor sign condition (3.1.7) to prove energy estimates for the system (3.1.1)-(3.1.5) in the case that $\mathcal{D}_t$ is a bounded domain, and later Lindblad [7] proved that this problem is locally well-posed in Sobolev spaces using a Nash-Moser iteration. The same result was later shown by Coutand-Shkoller [8] using a tangential smoothing operator as well as by [9] who used a more geometric approach which also applies on an unbounded domain.

Relatively little is known about the long-term behavior of solutions to the problem (3.1.1)-(3.1.5) with nonzero vorticity. We recall that in the case without free boundary and without gravity:

$$\partial_t + v^k \partial_k v_i + \partial_i p = 0 \quad \text{in } \mathbb{R}^3, \tag{3.1.8}$$

$$\operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3, \tag{3.1.9}$$

non-trivial vorticity is the obstacle to obtaining a global-in-time solution. By [10], if there are constants M_0, T_* so that if $T < T_*$ and $v \in C([0,T]; H^s(\mathbb{R}^3)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3))$ solves (3.1.8)-(3.1.9) and the a priori estimate:

$$\int_0^T ||\omega(s)||_{L^\infty(\mathbb{R}^3)}\, ds \leq M_0, \tag{3.1.10}$$

holds, then the solution can be extended to $v \in C([0,T^*]; H^s(\mathbb{R}^3)) \cap C^1([0,T^*]; H^s(\mathbb{R}^3))$. It then follows from the fact that:

$$(\partial_t + v^k \partial_k)\omega = \omega \cdot \partial v \tag{3.1.11}$$

and this result that if $\omega = 0$ at $t = 0$, sufficiently regular solutions to (3.1.8)-(3.1.9) can be extended to $T = \infty$. See also [11] for an extension to the case of a fixed domain with Neumann boundary condition and [12] for an extension to the free-boundary problem on a bounded domain.

In [13], the authors consider the Euler-Maxwell one-fluid system with nontrivial vorticity but without free boundary in three dimensions and proved a somewhat similar result. They prove that there is a norm $||\cdot||$ so that if $||\operatorname{curl} v(0,\cdot)|| \leq \delta$ for sufficiently small δ, then one can continue the solution up to $T \sim \delta^{-1}$. In particular, this provides a proof of global existence when $\operatorname{curl} v(0,\cdot) = 0$ for the Euler-Maxwell system.

Returning to the free boundary problem, to the best of our knowledge, the only papers that address the issue of the long-time behavior of solutions in the prescence of nontrivial vorticity are [14],[15] and [16]. In [14] Ifrim-Tataru prove that in two space dimensions (with one-dimensional boundary), solutions with constant vorticity can be continued up to $T \sim \varepsilon^{-2}$ if the initial data is of size ε. This is in constrast to the lifespan $T \sim \varepsilon^{-1}$ which is guaranteed by the local well-posedness theory. See also [15] in which Bieri-Miao-Shahshahani-Wu prove a similar result for a self-gravitating liquid occupying a bounded region. In [16], the authors consider the problem in arbitrary dimension and prove that the solution can be continued so long as the mean curvature of the boundary and $||\nabla v||_{L^\infty(\mathcal{D}_t)}$ are bounded.

For our result, we we will measure the regularity of ω in the norm:

$$||\omega(t)||^2_{H^r_w(\mathcal{D}_t)} = \sum_{k\leq r} \int_{\mathcal{D}_t} (1+|x|^2+y^2)^2 |\partial^k_{x,y}\omega(t,x,y)|^2\, dxdy, \tag{3.1.12}$$

and we will be considering solutions of Euler's equation with $\omega \cdot n|_{\partial\mathcal{D}_t} = 0$. Our main theorem is an analog of the result in [13]:

Theorem 3.1.1. *Fix $N_1 \geq 6$ and $N \gg 1$. Define $N_0 = 2NN_1$. There are constants $0 < \varepsilon_1^* \ll \varepsilon_0^* \ll 1$ satisfying the following property. Suppose that v_0, h_0 satisfy:*

$$||v_0||_{H^{N_0}(\mathcal{D}_0)} + ||h_0||_{H^{N_0}(\mathbb{R}^2)} \leq \varepsilon_0 \leq \varepsilon_0^*, \tag{3.1.13}$$

and that the Taylor sign condition (3.1.7) *holds at* $t = 0$*. Suppose in addition that* $\omega_0 = \text{curl}\, v_0$ *satisfies the bound:*

$$||\omega_0||_{H^{N_1}_w(\mathcal{D}_0)} \leq \varepsilon_1 \leq \varepsilon_1^*. \tag{3.1.14}$$

Let (v, h) *be the solution to* (3.1.1)-(3.1.5) *with initial data* v_0, h_0*. Let* T_ω *be the largest time so that* $(\omega \cdot n)|_{\partial\mathcal{D}_t} = 0$ *for* $0 \leq t \leq T_\omega$*. Then the problem* (3.1.1)-(3.1.3) *has a unique solution* (v, h) *with initial data* (v_0, h_0) *with* $v(t) \in H^{N_0}(\mathcal{D}_t)$, $h(t) \in H^{N_0}(\mathbb{R}^2)$ *for* $0 \leq t \leq T'_{\varepsilon_0,\varepsilon_1}$*, where:*

$$T'_{\varepsilon_0,\varepsilon_1} = C_N \min\left(\frac{\varepsilon_0^{1/3}}{\varepsilon_1^{1/3}}, \frac{1}{\varepsilon_0^N}, T_\omega\right), \tag{3.1.15}$$

for a constant C_N *depending only on N and* $||(-n \cdot \partial p_0)^{-1}||_{L^\infty(\partial\mathcal{D}_0)}$*.*

Here, p_0 is determined from v_0, h_0 by solving:

$$\Delta p_0 = -(\partial_i v_0^j)(\partial_j v_0^i), \quad \text{in } \mathcal{D}_0, \tag{3.1.16}$$

$$p_0 = 0, \qquad \text{on } \partial\mathcal{D}_0. \tag{3.1.17}$$

One simple way to ensure that the condition $(\omega \cdot n)|_{\partial\mathcal{D}_t} = 0$ holds for all time is to assume that $\omega_0|_{\partial\mathcal{D}_0} = 0$, since by the transport equation (3.1.11) it then follows that $\omega|_{\partial\mathcal{D}_t} = 0$ for $t > 0$ as well (see Lemma 3.5.80). We therefore have the following corollary:

Corollary 3.1.1. *With the same hypotheses as Theorem 3.1.1, suppose in addition that* $\omega_0|_{\partial\mathcal{D}_0} = 0$*. Then the solution* (v, h) *can be continued until:*

$$T_{\varepsilon_0,\varepsilon_1} = C_N \min\left(\frac{\varepsilon_0^{1/3}}{\varepsilon_1^{1/3}}, \frac{1}{\varepsilon_0^N}\right). \tag{3.1.18}$$

Note that Theorem 3.1.1 implies that if $\omega_0 = 0$, the solution can be continued until $T \sim \varepsilon_0^{-N}$. See also [17] for a similar lifespan bound for irrotational water waves on a periodic domain.

By the results [4], [3], comparing to the result in [13] one would expect to be able to take $T_{\varepsilon_0,\varepsilon_1} \sim \frac{1}{\varepsilon_1}$ which would in turn give a new proof of global existence in the irrotational case. The difference between that work and this one is that solutions to the linearization of the system (3.1.1)-(3.1.2) with zero vorticity decay at a rate $1/t$, while in [13], the authors consider the Euler-Maxwell system, for which solutions to the linearized system decay at a rate $1/t^{1+\beta}$ for small β.

We also remark that at the heuristic level, in the interior the vorticity satisfies an equation of the form $\partial_t W = W^2$ and if $|W(t=0)| \leq \varepsilon_1$, this equation has a lifespan of $\sim 1/W_0$ which is better than $\sim 1/W_0^{1/3}$. The reason we get a worse result than this is as follows. One can think of the equations on the boundary as being of the form:

$$(\partial_t + i\Lambda^{1/2})u = L(w) + Q_1(u,u) + Q_2(u,w) + Q_3(w,w) \tag{3.1.19}$$

where $\Lambda = |\nabla|$, w is a nonlocal function of the vorticity ω and Q_1, Q_2, Q_3 are nonlinearities depending on derivatives of their arguments. In particular, the vorticity enters *linearly* into the equations and this leads to a shorter lifespan. We hope to address both of these issues in future work. The assumption that $(\omega \cdot n)|_{\partial\mathcal{D}_t}$ for $t \geq 0$ is crucial here; as we will see in Section 3.3, this allows us to derive an equation for the evolution of the variables on the boundary with a good structure, which we will need in order to prove dispersive estimates. We note that a similar, but not identical, formulation of these equations appears in [18].

3.1.1 Outline of the proof

As in other works on the global behavior of solutions to dispersive equations, the result follows from a bootstrap argument, consisting of energy estimates to control the L^2-based norms and dispersive estimates to control the L^∞-based norms. We start with the energy estimates.

The system (3.1.1)-(3.1.3) has the following conserved quantity:

$$E_0(t) = \int_{\mathcal{D}_t} |v(t)|^2 dxdy + \int_{\mathbb{R}^2} |h(t)|^2 \, dS. \tag{3.1.20}$$

Here, we are writing $\mathcal{D}_t = \{(x,y) | x \in \mathbb{R}^2, y \leq h(t,x))\}$. In the case $\omega = 0$, one can use that the system (3.1.1)-(3.1.3) reduces to a Hamiltonian system on the boundary (see (3.3.1)-(3.3.2)) and this leads to higher-order energy estimates. Since we are considering the case $\omega \neq 0$, we prove energy estimates for the system (3.1.1)-(3.1.3) directly. These energy estimates are based on the estimates in [6], and we extend their approach to the case of an unbounded domain. (See also [19] where similar estimates were proved for the compressible Euler equations with free boundary in an unbounded domain)

The energies are of the form:

$$\mathcal{E}^r(t) = \int_{\mathcal{D}_t} Q(D^r v, D^r v) \, dxdy + \int_{\partial\mathcal{D}_t} |\overline{D}^{\,r-2}\theta|^2 (-n \cdot \partial p)^{-1} \, dS$$
$$+ \int_{\mathcal{D}_t} |D^{r-1}\omega|^2 \, dxdy \tag{3.1.21}$$

where D is the covariant derivative in $\mathcal{D}_t$, $\overline{D}$ is the covariant derivative on $\partial\mathcal{D}_t$ and θ is the second fundamental form of $\partial\mathcal{D}_t$; writing n for the unit normal to $\partial\mathcal{D}_t$ and $\Pi_i^j = \delta_i^j - n_i n^j$ for the projection to the tangent space at the boundary, it is given by:

$$\theta_{ij} = \Pi_i^k \Pi_j^\ell D_k n_\ell. \tag{3.1.22}$$

Here Q is a quadratic form which is the usual norm $Q(\beta, \beta) = |\beta|^2$ away from the boundary and which is the norm of the projection to the tangent space at the boundary when restricted to the boundary, $Q(\beta, \beta) = |\Pi\beta|^2$. See Section 3.5 for a precise definition. These energies appear to lose control over normal derivatives of v near the boundary, but it follows from the elliptic estimates in section 3.4 (see, in particular Lemma 3.5.2) that $\mathcal{E}^r$ controls $||v||^2_{H^r(\mathcal{D}_t)}$. In

Theorem 3.5.1, we prove that:

$$\frac{d}{dt}\mathcal{E}^r(t) \lesssim \mathcal{A}(t)\Big(\mathcal{E}^r(t) + \mathcal{A}(t)P(\mathcal{E}^{r-1}(t), ..., \mathcal{E}^0(t))\Big), \tag{3.1.23}$$

where P is a homogeneous polynomial with positive coefficients and $\mathcal{A}$ is given by:

$$\mathcal{A}(t) = ||Dv(t)||_{L^\infty(\mathcal{D}_t)} + ||\theta(t)||_{W^{2,\infty}(\partial\mathcal{D}_t)} + ||Dp(t)||_{L^\infty(\mathcal{D}_t)}$$

$$+ ||D^2p(t)||_{L^\infty(\partial\mathcal{D}_t)} + ||DD_tp||_{L^\infty(\partial\mathcal{D}_t)}. \tag{3.1.24}$$

We now turn to the more difficult task of proving dispersive estimates, and for this we will need to change variables. In $\mathcal{D}_t$, we write:

$$v = \nabla_{x,y}\psi + v_\omega, \quad \Delta_{x,y}\psi = 0, \tag{3.1.25}$$

with $n \cdot \partial\psi = v \cdot n$ on $\partial\mathcal{D}_t$ and where $\operatorname{curl} v_\omega = \omega$. We also write $\varphi = \psi|_{\partial\mathcal{D}_t}$. It will be important that the energies $\mathcal{E}^r$ control norms of φ, h and v_ω. To see why this is the case, note that $\theta_{ij} = (1 + |\nabla h|^2)^{-1/2}\nabla_i\nabla_j h$ for $i, j = 1, 2$ and that $||h||^2_{L^2(\mathbb{R}^2)}$ is bounded by the conserved energy, from which it follows that $||h||^2_{H^r(\mathbb{R}^2)} \lesssim \sum_{\ell \leq r} \mathcal{E}^\ell$. To control φ, we start with the observation that:

$$\int_{\partial\mathcal{D}_t} \varphi\mathcal{N}\varphi \, dS = ||\nabla_{x,y}\psi||^2_{L^2(\mathcal{D}_t)} \leq ||v||^2_{L^2(\mathcal{D}_t)}, \tag{3.1.26}$$

where $\mathcal{N}$ is the Dirichlet-to-Neumann map. The left hand side controls $||\Lambda^{1/2}\varphi||_{L^2(\partial\mathcal{D}_t)}$ where $\Lambda = |\nabla|$. To control higher derivatives, we could repeat this argument with φ replaced by $\nabla^r\varphi$ but this would require controlling the commutator $[\mathcal{N}, \nabla^r]$ which is nontrivial. Instead it will suffice for our purposes to use a slightly weaker version of the standard trace inequality (see (3.4.26)) and estimate:

$$||\nabla_x \psi||^2_{H^{r-1}(\partial\mathcal{D}_t)} \lesssim ||\nabla_{x,y}\psi||^2_{H^r(\mathcal{D}_t)} \lesssim ||v||^2_{H^r(\mathcal{D}_t)} + ||v_\omega||^2_{H^r(\mathcal{D}_t)} \lesssim \mathcal{E}^r, \tag{3.1.27}$$

where the estimate for $||v_\omega||_{H^r(\mathcal{D}_t)}$ follows from the elliptic estimates in Section 3.4, since curl $v_\omega = \omega$ and $v_\omega \cdot n|_{\partial\mathcal{D}_t} = 0$. See Proposition 3.5.2. To highest order, the left-hand side here controls $||\nabla_x \varphi||_{H^{r-1}(\mathbb{R}^2)}$.

With the L^2 estimates out of the way, we now want to prove L^∞ estimates for φ, h. In section 3.3, we derive a system satisfied by φ and h. This system is well-known in the case that $\omega = 0$ (see e.g. [20]). In the case $\omega \neq 0$, Castro and Lannes [18] derived a system for ω, φ, h and proved local well-posedness. We will use a system which is similar but not identical to theirs. To motivate our derivation, we recall the basic idea behind the "good unknown" introduced in [21]. We write $V^i = v^i|_{\partial\mathcal{D}_t}$, $i = 1,2$ and $B = v^3|_{\partial\mathcal{D}_t}$ as well as $U = V + \nabla h B$. [1] After restricting Euler's equation (3.1.1) to $\partial\mathcal{D}_t$ and using the boundary condition (3.1.5), V and B satisfy the following equations:

$$\hat{D}_t V = -a\nabla h, \tag{3.1.28}$$

$$\hat{D}_t B = a - 1, \tag{3.1.29}$$

where $a = (\partial_y p)|_{\partial\mathcal{D}_t}$ and $\hat{D}_t = \partial_t + V^1\partial_1 + V^2\partial_2$. In particular, we have:

$$\hat{D}_t(V + \nabla h B) = -\nabla h - \hat{D}_t \nabla h. \tag{3.1.30}$$

In the case $\omega = 0$, $V = \partial_x \psi\big|_{\partial\mathcal{D}_t}$ and $B = \partial_y \psi\big|_{\partial\mathcal{D}_t}$, so by the chain rule, we have:

$$\nabla_x \varphi(x) = (\nabla_x \psi)(x, h(x)) + \nabla_x h(x)(\nabla_y \psi)(x, h(x)) = V + \nabla h B, \tag{3.1.31}$$

[1] The good unknown used in [21] is actually given by $U = V + T_{\nabla h} B$ where T is Bony's paraproduct but it will suffice to use this simpler definition for our purposes.

with $\varphi(x) = \psi(x, h(x))$. Plugging this into (3.1.30) gives an evolution equation for $\nabla\varphi$. It turns out that in the irrotational case, after making this substitution (3.1.30), is of the form:

$$\partial_t \nabla\varphi = \nabla F(\varphi, h), \tag{3.1.32}$$

for a nonlinearity $F(\varphi, h)$ which also depends on the derivatives of φ, h. This leads to an equation for $\partial_t\varphi$.

When $\omega \neq 0$, we write $v = \nabla_{x,y}\psi + v_\omega$ in $\mathcal{D}_t$ and let $V^i_\omega = v^i_\omega|_{\partial\mathcal{D}_t}$ for $i = 1, 2$ and $B_\omega = v^3_\omega|_{\partial\mathcal{D}_t}$. Repeating the above calculation leads to an equation of the form:

$$\partial_t \nabla\varphi + \partial_t(V_\omega + \nabla h B_\omega) = \nabla F(\varphi, h) + G(\varphi, h, V_\omega, B_\omega), \tag{3.1.33}$$

with the same F as above. Writing $U_\omega = V_\omega + \nabla h B_\omega$, the crucial observation is that:

$$\operatorname{curl}_2 U_\omega = \partial_1 U^2_\omega - \partial_1 U^2_\omega = \omega \cdot n \qquad \text{on } \mathbb{R}^2. \tag{3.1.34}$$

See Theorem 3.3.1. In particular, if $\omega \cdot n = 0$ it follows that $U_\omega = \nabla a_\omega$ for a function a_ω. Making this substitution in (3.1.33), it turns out that G is a gradient, $G = \nabla H(\varphi, h, V_\omega, B_\omega)$ for some other nonlinearity H, and the system becomes:

$$\partial_t(\nabla\varphi + \nabla a_\omega) = \nabla\big(F(\varphi, h) + H(\varphi, h, V_\omega, B_\omega)\big), \tag{3.1.35}$$

which gives an evolution equation for $\varphi_\omega = \varphi + a_\omega$. Setting $u = h + i\Lambda^{1/2}\varphi_\omega$ and writing $w = (V_\omega, B_\omega)$ (3.1.35) and (3.1.4) lead to an equation of the form:

$$(\partial_t + i\Lambda)u = N(u) + L(w) + N_1(u, w) + N_2(w, w), \tag{3.1.36}$$

where N, N_1, N_2 are a nonlinear operators and L is linear. See Proposition 3.3.1 for the precise form of the right-hand side.

The nonlinearity N is the same one that occurs in [4], and can be handed using simple modifications of the arguments there. Specifically, we start with the Duhamel representation of the system (3.1.36):

$$e^{it\Lambda^{1/2}}u(t) = u_0 + \int_0^t e^{is\Lambda^{1/2}}N(u)\,ds + \int_0^t e^{is\Lambda^{1/2}}\Big(L(w) + N_1(u,w) + N_2(w,w)\Big)\,ds \quad (3.1.37)$$

$$\equiv u_0 + f_1(u) + f_2(u,w). \quad (3.1.38)$$

We follow [4] and define:

$$||u||_X = \sup_{0\le t\le T}(1+t)||u||_{W^{4,\infty}(\mathbb{R}^2)} + (1+t)^{-\delta}\big(||u||_{H^{N_0}(\mathbb{R}^2)} + ||\Lambda^{\iota}xe^{it\Lambda^{1/2}}u||_{L^2(\mathbb{R}^2)}\big), \quad (3.1.39)$$

where here ι, δ are sufficiently small constants.

Minor changes to the arguments in [4] (which we outline in Section 3.7) show that if $||u||_X \le \varepsilon_0$ and $\sup_{0\le\tau\le t}||\omega(\tau)||_{H_w^{N_1}(\mathcal{D}_\tau)} \le \varepsilon_1$, then:

$$(1+t)||e^{-it\Lambda^{1/2}}f_1(u)||_{W^{4,\infty}(\mathbb{R}^2)} \lesssim \varepsilon_0^2 + (1+t)^2\varepsilon_1, \quad (3.1.40)$$

$$(1+t)^{-\delta}||\Lambda^{\iota}xf_1(u)||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_0^2 + (1+t)^{2-\delta}\varepsilon_1. \quad (3.1.41)$$

In Section 3.6, we prove bounds of the above form for f_2, IE:

$$(1+t)||e^{-it\Lambda^{1/2}}f_2(u,w)||_{W^{4,\infty}(\mathbb{R}^2)} \lesssim \varepsilon_0^2 + (1+t)^2\varepsilon_1, \quad (3.1.42)$$

$$(1+t)^{-\delta}||\Lambda^{\iota}xf_2(u,w)||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_1^2 + (1+t)^{2-\delta}\varepsilon_1. \quad (3.1.43)$$

The proof of (3.1.42)-(3.1.43) requires bounding norms of $w = (V_\omega, B_\omega)$ on the boundary in terms of ω in the interior, for which we use the elliptic estimates in Section 3.4. These estimates combined with the above the above energy estimates and a continuity argument show that the solution can be continued until $T \sim T_{\varepsilon_0,\varepsilon_1}$.

3.2 Proof of the main theorem

We begin by decomposing our initial velocity v_0 into its irrotational and rotational parts. Given $h_0 : \mathbb{R}^2 \to \mathbb{R}$, set $\mathcal{D}_0 = \{(x_1, x_2, y) | y \le h_0(x_1, x_2)\}$. We now write $v_0 = \nabla_{x,y}\psi_0 + v_{\omega_0}$, where $\Delta\psi_0 = 0$ in $\mathcal{D}_0$, $\partial_n \psi_0 = v_0 \cdot n$ on $\partial\mathcal{D}_0$, and where $\operatorname{curl} v_{\omega_0} = \omega_0 \equiv \operatorname{curl} v_0$. We also write $V^i_{\omega_0} = v^i_{\omega_0}|_{\partial\mathcal{D}_0}, i = 1, 2$ and $B_{\omega_0} = v^3_{\omega_0}|_{\partial\mathcal{D}_0}$. In Section 3.3, we prove that if $\omega_0|_{\partial\mathcal{D}_0} = 0$, then $V_{\omega_0} + \nabla h_0 B_{\omega_0} = \nabla a_{\omega_0}$ for a function a_{ω_0}. We then write $\varphi_0 = \psi_0|_{\partial\mathcal{D}_0}$ as well as $\varphi_{\omega_0} = \varphi_0 + a_{\omega_0}$ and $u_0 = h_0 + i\Lambda^{1/2}\varphi_{\omega_0}$, where $\Lambda = |\nabla|$.

We now fix $N_1 \ge 6$, $N \gg 1$ and set $N_0 = 2NN_1$. With the above notation and with $||\cdot||_{H^{N_1}_w}$ defined by (3.1.12), we suppose that v_0, ω_0, h_0 satisfy:

$$||v_0||_{L^\infty(\mathcal{D}_t)} + ||v_0||_{H^{N_0}(\mathcal{D}_0)} + ||h_0||_{H^{N_0}(\mathcal{D}_0)}$$
$$+ ||u_0||_{W^{4,\infty}(\mathbb{R}^2)} + ||\Lambda^\iota x u_0||_{L^2(\mathbb{R}^2)} \le \frac{1}{2}\varepsilon_0, \quad (3.2.1)$$

$$\omega_0 \cdot n_0|_{\partial\mathcal{D}_0} = 0, \quad (3.2.2)$$

$$||\omega_0||_{H^{N_1}_w(\mathcal{D}_0)} \le \frac{1}{2}\varepsilon_1 \ll \varepsilon_0, \quad (3.2.3)$$

for sufficiently small ε_0 and ι, where n_0 is the unit normal to $\partial\mathcal{D}_0$.

We define $p_0 : \mathcal{D}_0 \to \mathbb{R}$ by:

$$\Delta p_0 = -(\partial_i v^j_0)(\partial_j v^i_0) \quad \text{in } \mathcal{D}_0, \quad (3.2.4)$$

$$p_0 = 0 \qquad \text{on } \partial\mathcal{D}_0. \quad (3.2.5)$$

In order for the initial value problem (3.1.1)-(3.1.6) to be well-posed, we need to ensure that $(-\nabla_{n_0} p_0) \ge \delta_0 > 0$ for some δ_0. In the irrotational case, this condition holds automatically by the Hopf lemma, because then $\Delta p_0 = -(\partial v)^2 \leq 0$ (see [22]). When $\operatorname{curl} v_0 \neq 0$ we instead have the following result:

Lemma 3.2.1. *Suppose that $||\omega_0||_{L^\infty(\mathcal{D}_0)} \leq \frac{1}{2}||v_0||_{L^\infty(\mathcal{D}_0)}$. Then, with p_0 defined by (3.2.5), there is a constant $c_0 > 0$ so that:*

$$(-\nabla_{n_0} p_0) \geq 2c_0 > 0 \text{ on } \partial\mathcal{D}_0. \tag{3.2.6}$$

Proof. We follow the argument in [22]. We fix a function $f : \partial\mathcal{D}_0 \to \mathbb{R}$ and let F denote its harmonic extension to $\mathcal{D}_0$. By Green's identity:

$$\int_{\partial\mathcal{D}_0} f\nabla_{n_0}(p_0 + y) - (p_0 + y)\nabla_{n_0} f = \int_{\mathcal{D}_0} \Delta(p_0 + y)F. \tag{3.2.7}$$

We now note that $\Delta p_0 = -(\partial_i v_0^j)(\partial_j v_0^i) = -(\partial_i v_0^j)(\partial_i v_0^j) + (\partial_i v_0^j)\delta^{i\ell}(\operatorname{curl} v_0)_{j\ell}$. By assumption we have that $||\partial v_0 - \operatorname{curl} \omega||_{L^\infty(\mathcal{D}_t)} \geq \frac{1}{2}||\partial v_0||_{L^\infty(\mathcal{D}_t)}$ and so in particular we have that $\Delta p_0 < 0$. Therefore by (3.2.7) and the fact that $p_0 = 0$ on $\partial\mathcal{D}_0$, we have:

$$\int_{\partial\mathcal{D}_0} fD_{n_0}(p_0 + y) - yD_{n_0} f > 0. \tag{3.2.8}$$

The rest of the proof of Lemma 4.1 from [22] now goes through without change. $\square$

We will use the following local well-posedness result, which follows from Theorem B in [9] and the above lemma:

Proposition 3.2.1. *Let $h_0 \in H^{N_0}(\mathbb{R}^2)$, $\mathcal{D}_0 = \{(x_1, x_2, y) | y \leq h_0(x_1, x_2)\}$ $v_0 \in H^{N_0}(\mathcal{D}_0)$. Suppose that $||\omega_0||_{L^\infty(\mathcal{D}_0)} \leq \frac{1}{2}||v_0||_{L^\infty(\mathcal{D}_0)}$. Then there is a $T = T(v_0, h_0) > 0$, a function $h : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and a vector field $v = v(t)$ defined on $\mathcal{D}_t \equiv \{(x, y) | y \leq h(t, x)\}$ for $0 \leq t \leq T$, so that $v\big|_{t=0} = v_0, \mathcal{D}_t\big|_{t=0} = \mathcal{D}_0$, $(v, \mathcal{D}_t)$ satisfy (3.1.1)-(3.1.5) and $v(t, \cdot) \in H^{N_0}(\mathcal{D}_t)$, $h(t, \cdot) \in H^{N_0}(\mathbb{R}^2)$ for $t \leq T$.*

We now want to extend the time T in this theorem to $T'_{\varepsilon_0,\varepsilon_1}$ defined in (3.1.15), provided that the vorticity ω vanishes on $\partial\mathcal{D}_t$. We suppose that u, v satisfy the following bootstrap assumptions for $t \geq 0$:

$$||u(t)||_{W^{4,\infty}(\mathbb{R}^2)} \leq \frac{\varepsilon_0}{1+t} \tag{3.2.9}$$

$$||v(t)||_{H^{N_0}(\mathcal{D}_t)} + ||h(t)||_{H^{N_0}(\mathbb{R}^2)} + ||\Lambda^\iota x e^{it\Lambda^{1/2}} u(t)||_{L^2(\mathbb{R}^2)} \leq \varepsilon_0(1+t)^\delta, \tag{3.2.10}$$

$$||\omega(t)||_{H^{N_1}_w(\mathcal{D}_t)} \leq \varepsilon_1, \tag{3.2.11}$$

and that $\omega \cdot n|_{\partial\mathcal{D}_t} = 0$. In Section 3.5.4 we show that:

$$||\Lambda^{1/2}\varphi_\omega||_{H^{N_0-1}(\mathbb{R}^2)} \lesssim ||v(t)||_{H^{N_0}(\mathcal{D}_t)} + ||h(t)||_{H^{N_0}(\partial\mathcal{D}_t)} + O(\varepsilon_0^2), \tag{3.2.12}$$

if (3.2.9)-(3.2.10) hold. In particular, the assumption (3.2.10) implies an estimate for $||u||_{H^{N_0-1}(\mathbb{R}^2)}$, a fact which is used several times in the proofs of the following theorems.

Recalling the definitions in (3.1.38) and writing $w = (V_\omega, B_\omega)$, we have:

Proposition 3.2.2. *Fix $T > 0$. Suppose that the bootstrap assumptions (3.2.9)-(3.2.11) hold and that $\omega \cdot n|_{\partial\mathcal{D}_t} = 0$ for $0 \leq t \leq T$. Suppose additionally that $||h(t)||_{W^{3/2,3}(\mathbb{R}^2)} \lesssim 1$ for $0 \leq t \leq T$. Then:*

$$||e^{-it\Lambda^{1/2}} f_1(u)||_{W^{4,\infty}(\mathbb{R}^2)} + ||e^{-it\Lambda^{1/2}} f_2(u,w)||_{W^{4,\infty}(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0^2}{1+t} + \varepsilon_1(1+t)^2 \tag{3.2.13}$$

$$||\Lambda^\iota x f_1(u)||_{L^2(\mathbb{R}^2)} + ||\Lambda^\iota x f_2(u,w)||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_0^2(1+t)^{2\delta} + \varepsilon_1(1+t)^2, \tag{3.2.14}$$

for $0 \leq t \leq T$.

The estimates for f_1 follow from simple modifications of the arguments in [4] and we outline the proof in Section 3.7. The estimates for f_2 can be found in Sections 3.6.1-3.6.3. We remark that the assumption on the size of $||h||_{H^{N_1}(\mathbb{R}^2)}$ is only needed for Proposition 3.4.3 and can be avoided.

The estimates (3.2.13)-(3.2.14) imply that there is a constant C_0 so that if:

$$t \leq T_0 \equiv C_0 \frac{\varepsilon_0^{1/2}}{\varepsilon_1^{1/2}}, \tag{3.2.15}$$

then:

$$(1+t)||u(t)||_{W^{4,\infty}(\mathbb{R}^2)} \leq \frac{1}{2}\varepsilon_0 + C_1\varepsilon_0^2 \leq \frac{3}{4}\varepsilon_0, \tag{3.2.16}$$

$$(1+t)^{-\delta}||\Lambda^t x e^{it\Lambda^{1/2}} u(t)||_{L^2(\mathbb{R}^2)} \leq \frac{1}{2}\varepsilon_0 + C_2\varepsilon_0^2 \leq \frac{3}{4}\varepsilon_0, \tag{3.2.17}$$

We need some estimates to control the size of ω, which we prove in Section 3.5.5:

Proposition 3.2.3. *If the assumptions (3.2.9)-(3.2.11) hold, then there is a constant C_N so that:*

$$||\omega(t)||^2_{H_w^{N_1}(\mathcal{D}_t)} \leq ||\omega(0)||^2_{H_w^{N_1}(\mathcal{D}_0)} + C_{N_1}\left(\varepsilon_0\varepsilon_1^2(1+t)^{1/N} + \varepsilon_1^3(1+t)\right). \tag{3.2.18}$$

In particular, if:

$$2C_N t \leq T_1 \equiv \min\left(\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_0^N}\right), \tag{3.2.19}$$

this implies that:

$$||\omega(t)||^2_{H_w^N(\mathcal{D}_t)} \leq \frac{3}{4}\varepsilon_1^2. \tag{3.2.20}$$

The last ingredient we need is an energy estimate for the entire system, which we prove in Section 3.5:

Proposition 3.2.4. *If $||v_0||_{H^{N_0}(\mathcal{D}_0)} + ||\omega_0||_{H^{N_0-1}(\mathcal{D}_0)} \leq \varepsilon_0/2$ and the bootstrap assumptions (3.2.9)-(3.2.11) hold, then with c_0 as in Lemma 3.2.1, there are constants $C^E_{N_0} = C^E_{N_0}(c_0)$ and K_{N_0} so that:*

$$
\begin{aligned}
||v(t)||^2_{H^{N_0}(\mathcal{D}_t)} + ||h(t)||^2_{H^{N_0}(\mathbb{R}^2)} \\
\leq \frac{\varepsilon_0^2}{4} + C^E_{N_0}\left(\varepsilon_0^3(1+t)^{2\delta} + \varepsilon_1\varepsilon_0^2(1+t)^{1+2\delta}\right) \\
+ C^E_{N_0}(\varepsilon_0(1+t)^{\delta})^{K_{N_0}}\varepsilon_1^2\varepsilon_0^2(1+t)^{1+2\delta}, \quad (3.2.21)
\end{aligned}
$$

if δ is sufficiently small.

We remark that a more careful energy estimate should show that the last term on the right-hand side of (3.2.21) could be made much smaller, but this will suffice for our purposes.

In particular, if t is such that:

$$
2C^E_{N_0} t \leq T_2 \equiv \min\left(\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_0^{1/\delta}}\right), \quad (3.2.22)
$$

this implies that, for ε_0 taken sufficiently small:

$$
||v(t)||^2_{H^{N_0}(\mathcal{D}_t)} + ||h(t)||^2_{H^{N_0}(\mathcal{D}_t)} \leq \frac{\varepsilon_0^2}{4} + \varepsilon_0^3(1+t)^{2\delta}. \quad (3.2.23)
$$

We now recall that we have taken δ so that $1/\delta < N_0 = 2NN_1$. Setting $T_{\varepsilon_0,\varepsilon_1} = \min(T_0, T_1, T_2)$, a standard continuity argument then gives Theorem 3.1.1.

3.3 Derivation of the equations on the boundary

We will use the equations (3.1.1)-(3.1.5) directly to prove energy estimates. However, to prove the dispersive estimates in Proposition 3.2.2, we will need to use equations for h and $v\big|_{\partial\mathcal{D}_t}$. In the irrotational case, $v_i = \partial_i\psi$ for a harmonic function ψ satisfying $n \cdot \partial\psi = v \cdot n$ on $\partial\mathcal{D}_t$.

Letting $\varphi = \psi|_{\partial \mathcal{D}_t}$, one can show that h, φ satisfy the system:

$$\partial_t h = G(h)\varphi, \tag{3.3.1}$$

$$\partial_t \varphi = -h - \frac{1}{2}|\nabla \varphi|^2 + \frac{(G(h)\varphi + \nabla h \cdot \nabla \varphi)^2}{2(1+|\nabla h|^2)}, \tag{3.3.2}$$

where $G(h)$ is the rescaled Dirichlet-to-Neumann map (see (3.3.7)) and we are writing $\nabla = (\partial_1, \partial_2)$. This system is derived from the fact that when $\omega = 0$, Euler's equations become:

$$\partial_i\big(\partial_t \psi + |\partial \psi|^2 + p + y\big) = 0, \qquad \text{in } \mathcal{D}_t. \tag{3.3.3}$$

See e.g. [20] or [23] for a derivation.

This no longer works when $\omega \neq 0$ and so another approach is needed. Our derivation of the equations on the boundary is partially based on the ideas in [21] (see in particular Section 4.1 there). We define:

$$V^i = v^i|_{\partial \mathcal{D}_t} \text{ for } i = 1, 2, \qquad B = v^3|_{\partial \mathcal{D}_t}. \tag{3.3.4}$$

In what follows we will write $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 0, 1, 2, 3$ with the convention that $x^0 = t$. We will also occasionally write $\partial_y = \partial_3$. We will also write ∇ for the derivative of quantites defined on $\mathbb{R}^2 \sim \partial \mathcal{D}_t$ and ∂ when differentiating quantities defined on $\mathcal{D}_t$. We now collect a few well-known and elementary identities. Given $f : \mathcal{D}_t \to \mathbb{R}$, write $F(x) = f(x, h(x)) = f|_{\partial \mathcal{D}_t}(x)$. Then, by the chain rule:

$$\partial_i F = (\partial_i f)|_{\partial \mathcal{D}_t} + \nabla_i h (\partial_y f)|_{\partial \mathcal{D}_t}, \quad i = 0, 1, 2 \tag{3.3.5}$$

If f is harmonic on $\mathcal{D}_t$ then additionally:

$$(\partial_y f)|_{\partial \mathcal{D}_t} = \frac{1}{1+|\nabla h|^2}\Big(G(h)F + \nabla h \cdot \nabla F\Big), \tag{3.3.6}$$

where $G(h)$ is the rescaled Dirichlet-to-Neumann operator:

$$G(h)F = \sqrt{1 + |\nabla h|^2} n \cdot \partial f|_{\partial \mathcal{D}_t}. \tag{3.3.7}$$

We also recall that the boundary condition (3.1.3) can be written:

$$\partial_t h + V^1 \nabla_1 h + V^2 \nabla_2 h = B. \tag{3.3.8}$$

As a consequence, writing $\hat{D}_t = \partial_t + V^1 \partial_1 + V^2 \partial_2$, we have:

$$\hat{D}_t F = (D_t f)|_{\partial \mathcal{D}_t}. \tag{3.3.9}$$

Next, in $\mathcal{D}_t$ we define $\psi \in L^6(\mathcal{D}_t) \cap \dot{H}^1(\mathcal{D}_t)$ to be the harmonic extension of $v \cdot n$ to $\mathcal{D}_t$, that is, ψ satisfies:

$$\Delta \psi = 0 \quad \text{in } \mathcal{D}_t, \quad n \cdot \partial \psi = n \cdot v \quad \text{on } \partial \mathcal{D}_t. \tag{3.3.10}$$

The function ψ is unique since $\operatorname{div} v = 0$ in $\mathcal{D}_t$.

It then follows that $v_\omega \equiv v - \partial \psi$ satisfies:

$$\operatorname{curl} v_\omega = \omega, \quad \operatorname{div} v_\omega = 0, \text{ on } \mathcal{D}_t, \quad n \cdot v_\omega = 0 \text{ on } \partial \mathcal{D}_t. \tag{3.3.11}$$

We write:

$$\varphi = \psi|_{\partial \mathcal{D}_t}, \quad V^i_\omega = v^i_\omega|_{\partial \mathcal{D}_t}, \, i = 1, 2, \quad B_\omega = v^3_\omega|_{\partial \mathcal{D}_t}. \tag{3.3.12}$$

The following derivation is inspired by the approach of [21] and [16]. The main result of this section is the following:

Theorem 3.3.1. *With the above notation:*

1. *Writing $U_\omega = V_\omega + \nabla h B_\omega$, we have $\nabla_1 U^2_\omega - \nabla_2 U^1_\omega = \omega|_{\partial \mathcal{D}_t} \cdot n$, In particular, $U_\omega = \nabla a_\omega$ for a function $a_\omega : \mathbb{R}^2 \to \mathbb{R}$ if $\omega|_{\partial \mathcal{D}_t} \cdot n = 0$.*

2. *The variables φ, h, V_ω and B_ω satisfy the system:*

$$\partial_t h = G(h)\varphi, \tag{3.3.13}$$

$$\partial_t(\varphi + a_\omega) = -h - \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\frac{(G(h)\varphi + \nabla h\cdot\nabla\varphi)^2}{1+|\nabla h|^2} + R_\omega, \tag{3.3.14}$$

where:

$$R_\omega = -\frac{1}{2}|V_\omega|^2 - \nabla\varphi\cdot V_\omega - \frac{1}{2}\big(V_\omega\cdot\nabla h\big)^2 + (G(h)\varphi)V_\omega\cdot\nabla h \tag{3.3.15}$$

Proof. By (3.3.5):

$$\nabla_1 U_\omega^2 - \nabla_2 U_\omega^1 = \nabla_1 V_\omega^2 - \nabla_2 V_\omega^1 + (\nabla_2 h)\nabla_1 B_\omega - (\nabla_1 h)\nabla_2 B_\omega \tag{3.3.16}$$

$$= (\partial_1 v_\omega^2)|_{\partial\mathcal{D}_t} - (\partial_2 v_\omega^1)|_{\partial\mathcal{D}_t} + (\nabla_1 h)(\partial_3 v_\omega^2)|_{\partial\mathcal{D}_t} - (\nabla_2 h)(\partial_3 v_\omega^1)|_{\partial\mathcal{D}_t} \tag{3.3.17}$$

$$+ (\nabla_2 h)(\partial_1 v_\omega^3)|_{\partial\mathcal{D}_t} - (\nabla_1 h)(\partial_2 v_\omega^3)|_{\partial\mathcal{D}_t} \tag{3.3.18}$$

$$+ (\nabla_2 h)(\nabla_1 h)(\partial_3 v_\omega^3)|_{\partial\mathcal{D}_t} - (\nabla_1 h)(\nabla_2 h)(\partial_3 v_\omega^3)|_{\partial\mathcal{D}_t} \tag{3.3.19}$$

$$= (\partial_1 v_\omega^2 - \partial_2 v_\omega^1)|_{\partial\mathcal{D}_t} \tag{3.3.20}$$

$$- \Big((\nabla_1 h)(\partial_2 v_\omega^3 - \partial_3 v_\omega^2)|_{\partial\mathcal{D}_t} + (\nabla_2 h)(\partial_3 v_\omega^1 - \partial_1 v_\omega^3)|_{\partial\mathcal{D}_t}\Big) \tag{3.3.21}$$

$$= \omega^3|_{\partial\mathcal{D}_t} - (\nabla_i h)\,\omega^i|_{\partial\mathcal{D}_t}, \tag{3.3.22}$$

which gives the first result.

We now derive (3.3.13) -(3.3.14). Differentiating (3.3.8) and using the fact that $[\hat{D}_t, \nabla] = -\nabla V^k \nabla_k$ gives:

$$\hat{D}_t \partial_i h = \nabla_i B - \nabla_i V^k \nabla_k h \tag{3.3.23}$$

Writing $a = (\partial_y p)\big|_{\partial\mathcal{D}_t}$, restricting Euler's equations (3.1.1) to the boundary and using that $p = 0$ on $\partial\mathcal{D}_t$ gives:

$$\hat{D}_t V_i = -a\nabla_i h, \quad i = 1, 2 \tag{3.3.24}$$

$$\hat{D}_t B = a - 1. \tag{3.3.25}$$

Therefore:

$$\hat{D}_t(\nabla\varphi + U_\omega) = \hat{D}_t(V + \nabla h B) = -\nabla h + (\hat{D}_t \nabla h)B \tag{3.3.26}$$

$$= -\nabla h + (\nabla B - \nabla V^k \nabla_k h)B \tag{3.3.27}$$

$$= -\nabla h + \frac{1}{2}\nabla|B|^2 - \nabla V^k \nabla_k h B. \tag{3.3.28}$$

We will write $f_b = f\big|_{\partial\mathcal{D}_t}$ for the restriction to the boundary. Expanding out the definition of $\hat{D}_t$ and recalling by convention, sums over repeated upper and lower indices run over only the first two indices gives that:

$$\hat{D}_t(\nabla\varphi + U_\omega) = \partial_t(\nabla\varphi + U_\omega) + (\partial_k\psi)_b \nabla^k \nabla\varphi + (\partial_k\psi)_b \nabla^k U_\omega + V_\omega^k \nabla_k \nabla\varphi + V_\omega^k \nabla_k U_\omega \tag{3.3.29}$$

Combining this with (3.3.28) and expanding $(\partial_k\psi)_b = \nabla_k\varphi - \nabla^k h(\partial_y\psi)_b$, we have:

$$\partial_t(\nabla\varphi + U_\omega) = -\nabla h + \frac{1}{2}\nabla|B|^2 - \nabla V^k \nabla_k h B - \big(\nabla^k\varphi - (\partial_y\psi)_b \nabla^k h\big)\nabla_k \nabla\varphi \tag{3.3.30}$$

$$- \nabla^k\varphi \nabla_k U_\omega + (\partial_y\psi)_b \nabla^k h \nabla_k U_\omega - V_\omega^k \nabla_k \nabla\varphi - V_\omega^k \nabla_k U_\omega. \tag{3.3.31}$$

Expanding V, B in terms of ψ and v_ω and using (3.3.5):

$$\nabla V^k \nabla_k h B = \nabla(\partial_k \psi)_b \nabla^k h(\partial_y \psi)_b + \nabla V_\omega^k \nabla_k h B_\omega + \nabla(\partial_k \psi)_b \nabla^k h B_\omega + \nabla V_\omega^k \nabla_k h(\nabla_y \psi)_b \tag{3.3.32}$$

$$= (\nabla \nabla_k \varphi)\nabla^k h(\partial_y \psi)_b + \nabla V_\omega^k \nabla_k h B_\omega + \nabla(\nabla^k \varphi)\nabla_k h B_\omega \tag{3.3.33}$$

$$- \nabla(\nabla^k h(\partial_y \psi)_b)\nabla_k h B_\omega - \nabla(\nabla_k h(\partial_y \psi)_b)\nabla^k h(\partial_y \psi)_b + \nabla V_\omega^k \nabla_k h(\partial_y \psi)_b \tag{3.3.34}$$

We insert this expression into the previous one to get:

$$\partial_t(\nabla\varphi + U_\omega) = A(\varphi, h) + \frac{1}{2}\nabla|B_\omega|^2 + \nabla((\partial_y \psi)_b B_\omega) \tag{3.3.35}$$

$$- \nabla V_\omega^k \nabla_k h B_\omega - V_\omega^k \nabla_k U_\omega \tag{3.3.36}$$

$$- \nabla\nabla^k \varphi(\nabla_k h B_\omega) - \nabla^k \varphi \nabla_k U_\omega - V_\omega^k \nabla_k \nabla\varphi \tag{3.3.37}$$

$$+ (\partial_y \psi)_b \nabla^k h \nabla_k U_\omega + \nabla(\nabla^k h(\partial_y \psi)_b)\nabla_k h B_\omega - \nabla V_\omega^k \nabla_k h(\partial_y \psi)_b, \tag{3.3.38}$$

where A is given by:

$$A = -\nabla h + \frac{1}{2}\nabla(\partial_y \psi)_b^2 - \frac{1}{2}\nabla|\nabla\varphi|^2 + (\partial_y \psi)_b \nabla^k h \nabla_k \nabla\varphi \tag{3.3.39}$$

$$- \Big(\nabla(\nabla^k \varphi)\nabla_k h + \nabla(\nabla^k h(\partial_y \psi)_b)\Big)(\partial_y \psi)_b \tag{3.3.40}$$

$$= -\nabla h + \frac{1}{2}\nabla(\partial_y \psi)_b^2 - \frac{1}{2}\nabla|\nabla\varphi|^2 - \nabla\big(\nabla^k h(\partial_y \psi)_b\big)(\partial_y \psi)_b, \tag{3.3.41}$$

using (3.3.5) in the last step. Applying (3.3.6) shows that:

$$A = \nabla\bigg(- h - \frac{1}{2}|\nabla\varphi|^2 + \frac{(G(h)\varphi + \nabla h \cdot \nabla\varphi)^2}{2(1 + |\nabla h|^2)}\bigg) \tag{3.3.42}$$

We now want to show that all of the other terms in (3.3.38) are also gradients.

To handle the terms on the second row of (3.3.38), we note that by the definition of U_ω:

$$\nabla V_\omega^k \nabla_k h B_\omega + V_\omega^k \nabla_k U_\omega = \nabla V_\omega^k (\delta_{k\ell} U_\omega^\ell) + V_\omega^k \nabla_k U_\omega - \nabla V_\omega^k (\delta_{k\ell} V_\omega^\ell)$$

$$= \nabla(\delta_{k\ell} V_\omega^k U_\omega^\ell) - \frac{1}{2}\nabla |V_\omega|^2 \quad (3.3.43)$$

where we used the fact that curl $U_\omega = 0$ in the last step.

To deal with the terms on the third row of (3.3.38) we note that:

$$\nabla_k \varphi \nabla^k U_\omega + V_\omega^k \nabla_k \nabla \varphi + \nabla \nabla_k \varphi \nabla^k h B_\omega$$

$$= \nabla_k \varphi \nabla^k U_\omega + \nabla_k \nabla \varphi (V_\omega^k + \nabla^k h B_\omega) = \nabla(\nabla_k \varphi U_\omega^k). \quad (3.3.44)$$

Finally, to handle the terms on the last line of (3.3.38), we again use that curl $U_\omega = 0$ and expand out $U_\omega = V_\omega + \nabla h B_\omega$ and write the result as:

$$(\partial_y \psi)_b \big(\nabla_k h \nabla_i U_\omega^k + \nabla_i \nabla^k h \nabla_k h B_\omega - \nabla_i V_\omega^k \nabla_k h\big) + |\nabla h|^2 B_\omega \nabla_i (\partial_y \psi)_b \quad (3.3.45)$$

$$= (\partial_y \psi)_b \big(\nabla_k h \nabla_i V_\omega^k + \nabla_k h \nabla_i (\nabla_k h B_\omega) + \nabla_i \nabla_k h \nabla^k h B_\omega - \nabla_i V_\omega^k \nabla_k h\big) \quad (3.3.46)$$

$$+ |\nabla h|^2 B_\omega \nabla_i (\partial_y \psi)_b \quad (3.3.47)$$

$$= \nabla_i \big(|\nabla h|^2 (\partial_y \psi)_b B_\omega\big). \quad (3.3.48)$$

Combining the results of (3.3.43)-(3.3.48), we see that (3.3.38) becomes:

$$\partial_t (\nabla \varphi + \nabla a_\omega) = A(\varphi, h)$$

$$+ \frac{1}{2}\nabla |B_\omega|^2 + \frac{1}{2}\nabla |V_\omega|^2 - \nabla\big(V_\omega \cdot U_\omega\big) + \nabla((1 + |\nabla h|^2)(\partial_y \psi)_b B_\omega) - \nabla(\nabla \varphi \cdot U_\omega) \quad (3.3.49)$$

Now we note that since $v_\omega \cdot n = 0$, we have $B_\omega = V_\omega^k \nabla_k h$ which further implies $U_\omega = V_\omega + \nabla h(V_\omega \cdot \nabla h)$. The second line of (3.3.49) then becomes the gradient of:

$$-\frac{1}{2}(\nabla h \cdot V_\omega)^2 - \frac{1}{2}(V_\omega)^2 + \Big((1+|\nabla h|^2)(\partial_y \psi)_b - \nabla\varphi \cdot \nabla h\Big)(\nabla h \cdot V_\omega) - \nabla\varphi \cdot V_\omega$$
$$= -\frac{1}{2}(\nabla h \cdot V_\omega)^2 - \frac{1}{2}(V_\omega)^2 + (G(h)\varphi)(V_\omega \cdot \nabla h) - \nabla\varphi \cdot V_\omega \quad (3.3.50)$$

Next, we note that the vorticity does not enter into h (3.3.8) when we write v in terms of ψ, v_ω. Indeed, recalling that $v_\omega \cdot n = 0$ on $\partial\mathcal{D}_t$ and using (3.3.8) gives:

$$\partial_t h = -(\partial\psi)_b \cdot \nabla h - V_\omega \cdot \nabla h + (\partial_y \psi)_b + B_\omega \quad (3.3.51)$$

$$= \sqrt{1+|\nabla h|^2}\big((\partial\psi)_b \cdot n + (v_\omega)_b \cdot n\big) \quad (3.3.52)$$

$$= G(h)\varphi, \quad (3.3.53)$$

where in the last step we used that $\Delta\psi = 0$ in $\mathcal{D}_t$.

Combining the result of the above calculation with (3.3.53) completes the proof. $\square$

It is a little awkward to work in terms of a_ω, since it depends on the vorticity in the interior in a complicated way, and moreover we only control ∇a_ω, not a_ω itself. For this reason we set:

$$\varphi_\omega = \varphi + a_\omega. \quad (3.3.54)$$

The above system becomes:

$$\partial_t h = G(h)\varphi_\omega - G(h)a_\omega, \quad (3.3.55)$$

$$\partial_t \varphi_\omega = -h - |\nabla\varphi_\omega|^2 + \frac{(G(h)\varphi_\omega + \nabla h \cdot \nabla\varphi_\omega)^2}{1+|\nabla h|^2} + \widetilde{R_\omega}, \quad (3.3.56)$$

where:

$$\widetilde{R_\omega} = -\frac{1}{2}|\nabla a_\omega|^2 + \nabla\varphi_\omega \cdot \nabla a_\omega + (1+|\nabla h|^2)^{-1}\Big((G(h)a_\omega + \nabla h \cdot \nabla a_\omega)^2$$

$$- 2(G(h)a_\omega + \nabla h \cdot \nabla a_\omega)(G(h)\varphi_\omega + \nabla h \cdot \nabla \varphi_\omega)\Big) - \frac{1}{2}|V_\omega|^2 - \nabla\varphi_\omega \cdot V_\omega$$

$$- \frac{1}{2}\big(V_\omega \cdot \nabla h\big)^2 + (G(h)\varphi_\omega)V_\omega \cdot \nabla h + \nabla a_\omega \cdot V_\omega - (G(h)a_\omega)V_\omega \cdot \nabla h) \quad (3.3.57)$$

We now recall that $\nabla a_\omega = V_\omega + \nabla h B_\omega$ and that $B_\omega = \nabla h \cdot V_\omega$. Writing $G(h)a_\omega = G(h)\Lambda^{-1}R \cdot \nabla a_\omega$, we note that V_ω enters *linearly* into these equations, since:

$$\partial_t h = G(h)\varphi_\omega - G(h)(\Lambda^{-1}R \cdot V_\omega) - G(h)(\Lambda^{-1}R \cdot (\nabla h \cdot V_\omega)). \quad (3.3.58)$$

We also note that V_ω, B_ω enter no more than quadratically into the remaining terms.

Using these identities, can further re-write:

$$\widetilde{R_\omega} = -|V_\omega \cdot \nabla h|^2 + (\nabla\varphi_\omega \cdot \nabla h)(V_\omega \cdot \nabla h)$$

$$+ G(h)\Big[\varphi_\omega - \Lambda^{-1}R \cdot V_\omega - \Lambda^{-1}R \cdot (\nabla h B_\omega)\Big](V_\omega \cdot \nabla h)$$

$$+ (1+|\nabla h|^2)^{-1}\Bigg(\Big(G(h)[\Lambda^{-1}R \cdot V_\omega] + \nabla h \cdot V_\omega\Big)^2$$

$$- (G(h)\Lambda^{-1}R \cdot V_\omega + \nabla h \cdot V_\omega)(G(h)\varphi_\omega + \nabla h \cdot \nabla\varphi_\omega)\Bigg)$$

$$+ \text{ more nonlinear terms} \quad (3.3.59)$$

We now recall the following expansion of $G(h)$ in powers of h:

$$G(h) = \Lambda + G_2(h) + G_3(h) + G_4(h), \quad (3.3.60)$$

with:

$$G_2(h) = -\nabla \cdot (h\nabla) + \Lambda(h\Lambda), \tag{3.3.61}$$

$$G_3(h) = \Lambda(h^2\Lambda^2) + \Lambda^2(h^2\Lambda) - 2(h\Lambda(h\Lambda)), \tag{3.3.62}$$

and where $G_4(h) \equiv G(h) - \Lambda - G_1(h) - G_2(h)$ vanishes to order 3 when $h = 0$. See [20] for a formal derivation of this expansion, and e.g. Appendix F of [4] for rigorous estimates for G_4. Here, we are using the notation:

$$\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \mathcal{F} f), \quad s \in \mathbb{R}, \tag{3.3.63}$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^2$.

In particular, keeping track of just the terms which are linear or quadratic, the above equations become:

$$\partial_t h = \Lambda\varphi_\omega - R \cdot V_\omega - \nabla \cdot (h\nabla\varphi_\omega) - \Lambda(h\Lambda\varphi_\omega) + \Lambda(hR \cdot V_\omega) + \nabla \cdot (hV_\omega) + ... \tag{3.3.64}$$

$$\partial_t \varphi_\omega = -h - |\nabla\varphi_\omega|^2 + (\Lambda\varphi_\omega)^2 + |R \cdot V_\omega|^2 + (R \cdot V_\omega)\Lambda\varphi_\omega + ... \tag{3.3.65}$$

We now set:

$$u = h + i\Lambda^{1/2}\varphi_\omega \tag{3.3.66}$$

With this definition, we can recover h, φ_ω from u:

$$h = \operatorname{Re} u, \qquad \varphi_\omega = \Lambda^{-1/2} \operatorname{Im} u. \tag{3.3.67}$$

In what follows, we will write $u_R = \operatorname{Re} u$ and $u_I = \operatorname{Im} u$. We will also write R_i for the

Riesz transform:

$$\mathcal{F}(R_i f)(\xi) = \frac{\xi_i}{|\xi|}(\mathcal{F}f)(\xi), \qquad i = 1, 2. \tag{3.3.68}$$

Proposition 3.3.1. *With the above definitions, we have:*

$$(\partial_t + i\Lambda^{1/2})u = N(u) + L(V_\omega) + N_1(u, V_\omega) + N_2(u, V_\omega) + N_3(u, V_\omega), \tag{3.3.69}$$

where $N(u) = B(u) + T(u) + R(u)$ *and:*

$$B(u) = \Lambda u_R(\Lambda^{1/2}u_I) + \nabla \cdot (u_R(\Lambda^{-1/2}\nabla u_I)) + i\Lambda^{1/2}\Big(|\Lambda^{-1/2}\nabla u_I|^2 + |\Lambda^{1/2}u_I|^2\Big) \tag{3.3.70}$$

$$T(u) = -\frac{1}{2}\Lambda(u_R^2\Lambda^{3/2}u_I) + \Lambda^2(u_R^2\Lambda^{1/2}u_I) - 2\Lambda(u_R\Lambda(u_R\Lambda^{1/2}u_I)) \tag{3.3.71}$$

$$+ i\Lambda^{1/2}\Big(\Lambda^{1/2}u_I\big(u_R\Lambda^{3/2}u_I - \Lambda(u_R\Lambda^{1/2}u_I)\big)\Big) \tag{3.3.72}$$

$$L(V_\omega) = -R \cdot V_\omega, \tag{3.3.73}$$

$$N_1(u, V_\omega) = \Lambda^{1/2}(R \cdot V_\omega \Lambda^{1/2}u_I) - \nabla \cdot (u_R V_\omega) + \Lambda(u_R R \cdot V_\omega) \tag{3.3.74}$$

$$N_2(V_\omega, V_\omega) = \Lambda^{1/2}(R \cdot V_\omega)^2, \tag{3.3.75}$$

and where $R(u)$ *(resp.* $N_3(u, V_\omega)$*) vanish to order 4 (resp. 3) when* $h = 0$*, and where* $N_3(u, V_\omega)$ *is quadratic in* V_ω *and its derivatives.*

For later use, we record the Duhamel form of these equations:

$$e^{it\Lambda^{1/2}}u(t) - u_0 = g_1(t) + g_2(t) + g_3(t) + g_4(t) + g_5(t), \tag{3.3.76}$$

where:

$$g_1(t) = \int_0^t e^{is\Lambda^{1/2}} N(u)\, ds, \tag{3.3.77}$$

$$g_2(t) = \int_0^t e^{is\Lambda^{1/2}} L(w)\, ds, \qquad g_3(t) = \int_0^t e^{is\Lambda^{1/2}} N_1(u, w)\, ds, \tag{3.3.78}$$

$$g_4(t) = \int_0^t e^{is\Lambda^{1/2}} N_2(w, w)\, ds, \qquad g_5(t) = \int_0^t e^{is\Lambda^{1/2}} N_3(u, w)\, ds, \tag{3.3.79}$$

3.4 Elliptic estimates and the regularity of the free boundary

Much of the material in the following sections is based heavily on the estimates and ideas in [6]. In [6], the authors consider the free boundary problem for a bounded fluid region, but extending their approach to the case of an unbounded domain is straightforward.

It is convenient to work in terms of Lagrangian coordinates, which we now define. We let Ω denote the lower half-plane in $\mathbb{R}^3$. In this section, we will use the convention that points in $\mathcal{D}_t$ are denoted by x and points in Ω are denoted by y. The Lagrangian coordinates $x(t) : \Omega \to \mathcal{D}_t$ are then defined by:

$$\frac{d}{dt} x_i(t, y) = v_i(t, x(t, y)) \quad y \in \Omega, \tag{3.4.1}$$

$$x(0, y) = y. \tag{3.4.2}$$

In these coordinates, the material derivative $D_t = \partial_t + v^k \partial_k$ becomes the usual time derivative:

$$D_t = \frac{\partial}{\partial t}\bigg|_{y=const.} = \frac{\partial}{\partial t}\bigg|_{x=const.} + v^k \frac{\partial}{\partial x_k}. \tag{3.4.3}$$

The Lagrangian coordinates x induce a time dependent (co)metric g on Ω:

$$g_{ab} = \delta_{ij} \frac{dx^i}{dy^a} \frac{dx^j}{dy^b}, \quad g^{ab} = \delta^{ij} \frac{dy^a}{dx^i} \frac{dy^b}{dx^j} \tag{3.4.4}$$

We use the convention that indices $a, b, c...$ denote quantities expressed in Lagrangian coordinates and indices $i, j, k, ..$ denote quantities expressed in the x coordinates. We let D denote the covariant derivative on Ω with respect to the metric g. We write Γ^a_{bc} for the Christoffel symbols:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\left(\frac{\partial}{\partial y^a} g_{bd} + \frac{\partial}{\partial y^b} g_{ad} - \frac{\partial}{\partial y^d} g_{ab}\right), \tag{3.4.5}$$

and the covariant derivative of a $(0, r)$ tensor β is then:

$$D_a \beta_{a_1 \cdots a_r} = \partial_{y^a} \beta_{a_1 \cdots a_r} - \Gamma^d_{aa_1} \beta_{da_2 \cdots a_r} - \cdots \Gamma_{aa_r} \beta_{a_1 \cdots a_{r-1} d}. \tag{3.4.6}$$

We let $d = d(t, p) = \mathrm{dist}_g(p, \partial\Omega)$ denote the geodesic distance with respect to the metric g from $p \in \Omega$ to $\partial\Omega$, and we define the unit normal to $\partial\Omega$ by:

$$n_a = \partial_a d, \qquad\qquad n^a = g^{ab} n_b. \tag{3.4.7}$$

We will also write n_i for the normal expressed in Eulerian coordinates:

$$n_i = \frac{\partial y^a}{\partial x^i} n_a, \qquad\qquad n^i = \delta^{ij} n_j \tag{3.4.8}$$

We let $\iota_0 = \iota_0(t)$ denote the injectivity radius of $\partial\mathcal{D}_t$ By definition, this is the largest number ι_0 so that the map:

$$(x, \iota) \to x + \iota n(x), \quad x \in \partial\mathcal{D}_t \tag{3.4.9}$$

is injective from $\partial\mathcal{D}_t \times (-\iota_0, \iota_0) \to \{x \in \mathcal{D}_t : d(t, p) < \iota_0\}$.

The (co)metric on $\partial\Omega$ is given by:

$$\gamma_{ab} = g_{ab} - n_a n_b, \qquad\qquad \gamma^b_a = \delta^b_a - n_a n^b, \tag{3.4.10}$$

and the second fundamental form of $\partial\Omega$ is:

$$\theta_{ab} = \gamma^c_a \gamma^d_b \nabla_c n_d. \tag{3.4.11}$$

We note that on $\partial\Omega$, if $\overline{D}$ denotes the covariant derivative on $\partial\Omega$ with respect to the metric γ, then:

$$\overline{D}_a\beta_{a_1\cdots a_r} = \gamma_a^b\gamma_{a_1}^{b_1}\cdots\gamma_{a_r}^{b_r}D_b\beta_{b_1\cdots b_r}. \tag{3.4.12}$$

In particular this implies that if q is a function on Ω with $q = 0$ on $\partial\Omega$ then $\gamma_a^b D_b q = 0$ on $\partial\Omega$.

3.4.1 The extension of the normal to the interior

Since d is the geodesic distance, we have $D_{Dd}Dd = 0$ and so $DDd = \tilde{\theta}$, where $\tilde{\theta}$ is the second fundamental form for the surfaces $\{d = \text{const}\}$. We will also write θ for the second fundamental form of $\partial\Omega$; if n_a is the unit normal vector to $\partial\Omega$, then:

$$\theta_{ab} = (\delta_a^c - n_a n^c)(\delta_b^d - n_b n^d)D_c n_d. \tag{3.4.13}$$

We now define an extension of the normal to a neighborhood of the boundary. We fix d_0 with $\iota_0/16 \le d_0 \le \iota_0/2$ and let $\eta \in C^\infty(\mathbb{R})$ be a function with $\eta(s) = 1$ when $|s| \le 1/2, \eta(s) = 0$ when $|s| \ge 3/4$, $0 \le \eta(s) \le 1$ and $|\eta'| \le 4$. We then define:

$$\tilde{n}_a(p) = \eta\left(\frac{d(p)}{d_0}\right)D_a d(x,y). \tag{3.4.14}$$

Close to the boundary, we have $\tilde{n}_a = D_a d$ and away from the boundary, $\tilde{n}_a = 0$. We will not need the following lemma explicitly but it is useful to note that we can control the regularity of $\tilde{n}$. See Lemma 3.10 in [6] for the proof.

Lemma 3.4.1. *With the above definitions, for each $y \in \partial\Omega$, if $d \le \iota_0/2$:*

$$|D\tilde{n}(q,d)| \le 2|\theta(q)|, \qquad |D_t\tilde{n}(q,d)| \le 6||h||_{L^\infty(\Omega)}, \tag{3.4.15}$$

where $h_{ab} = \frac{1}{2}D_t g_{ab}$.

We now extend γ to the interior of Ω. Abusing notation, we will write:

$$\gamma_{ab} = g_{ab} - \tilde{n}_a\tilde{n}_b, \qquad \gamma_a^b = \delta^{ac}\gamma_{bc}, \qquad \gamma^{ab} = g^{ac}g^{bd}\gamma_{cd}. \tag{3.4.16}$$

On $\partial\Omega$, γ_{ab} (resp. γ^{ab}) is just the metric (resp. cometric) on $\partial\Omega$ induced by g, and γ^a_b is the projection to $T(\partial\Omega)$. Away from $\partial\mathcal{D}_t$, $\gamma_{ab} = g_{ab}$ and γ^a_b is the identity map. The estimates in Lemma 3.4.1 then imply (see Lemma 3.11 in [6]):

Lemma 3.4.2. *With the above definitions, we have:*

$$|D\gamma| \leq C\bigg(||\theta||_{L^\infty(\Omega)} + \frac{1}{\iota_0}\bigg), \qquad |D_t\gamma| \leq C||h||_{L^\infty(\partial\Omega)} \tag{3.4.17}$$

3.4.2 Elliptic estimates

For notational convenience, in this section we write $x_3 = y$. We will use multi-index notation and write $I = (i_1, \ldots, i_r)$. We will write D^r for the operator which has components:

$$D^r_I = D_{i_1} \cdots D_{i_r}, \tag{3.4.18}$$

If $i_j = 1, 2$ for each $j = 1, ..., r$, we will also write ∇^r for the operator:

$$\nabla^r_I = \nabla_{i_1} \cdots \nabla_{i_r}. \tag{3.4.19}$$

We will also write:

$$\gamma^I_J = \gamma^{i_1}_{j_1} \cdots \gamma^{i_r}_{j_r}, \tag{3.4.20}$$

Let β be a $(0, r+1)$ tensor with $\beta_{i_1\cdots i_r i} = D^r_{i_1\cdots i_r}\alpha_i$ for some $(0,1)$-tensor α. We write:

$$(\operatorname{div}\beta)_I = \delta^{ij}D_j\beta_I = D^r_I(\delta^{ij}D_j\alpha_i), \tag{3.4.21}$$

$$(\operatorname{curl}\beta)_{ij} = D_i\beta_{Ij} - D_j\beta_{Ii} = D^r_I(D_i\alpha_j - D_j\alpha_i). \tag{3.4.22}$$

We will also write:

$$(\Pi\beta)_I = \gamma_I^J \beta_J, \tag{3.4.23}$$

$$(n\cdot\beta)_I = n^i\beta_{Ii} \tag{3.4.24}$$

We will rely heavily on the following pointwise estimate in $\mathcal{D}_t$, which is originally from [6]:

Lemma 3.4.3. *If β is as above, then:*

$$|D\beta|^2 \le C\big(\delta^{ij}\gamma^{k\ell}\gamma^{IJ}(D_k\beta_{Ii})(D_\ell\beta_{Jj}) + |\operatorname{div}\beta|^2 + |\operatorname{curl}\beta|^2\big), \qquad \textit{in } \mathcal{D}_t. \tag{3.4.25}$$

We will also use the following L^2 estimates:

Lemma 3.4.4. *With the above notation, if $|\theta| + \frac{1}{\iota_0} \le K$ then:*

$$||\beta||^p_{L^p(\partial\Omega)} \le C\big(||D\beta||_{L^p(\Omega)} + K||\beta||_{L^p(\Omega)}\big), \quad 1 < p < \infty, \tag{3.4.26}$$

$$||\beta||^2_{L^2(\partial\Omega)} \le C||\Pi\beta||^2_{L^2(\partial\Omega)} + C\big(||\operatorname{div}\beta||_{L^2(\Omega)} + ||\operatorname{curl}\beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\big)||\beta||_{L^2(\Omega)}, \tag{3.4.27}$$

$$||\beta||^2_{L^2(\partial\Omega)} \le C||n\cdot\beta||^2_{L^2(\partial\Omega)} + C\big(||\operatorname{div}\beta||_{L^2(\Omega)} + ||\operatorname{curl}\beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\big)||\beta||_{L^2(\Omega)}, \tag{3.4.28}$$

and

$$||D\beta||^2_{L^2(\Omega)} \le C||D\beta||_{L^2(\partial\Omega)}||\beta||_{L^2(\partial\Omega)} + C\big(||\operatorname{div}\beta||_{L^2(\Omega)} + ||\operatorname{curl}\beta||_{L^2(\Omega)}\big)^2, \tag{3.4.29}$$

$$\begin{aligned}||D\beta||^2_{L^2(\Omega)} \le C||\Pi D\beta||_{L^2(\partial\Omega)}||\Pi n\cdot\beta||_{L^2(\partial\Omega)}\\ + C\big(||\operatorname{div}\beta||_{L^2(\Omega)} + ||\operatorname{curl}\beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\big)^2,\end{aligned} \tag{3.4.30}$$

$$||D\beta||^2_{L^2(\Omega)} \leq C||\Pi n \cdot \nabla \beta||_{L^2(\partial\Omega)}||\Pi\beta||_{L^2(\partial\Omega)}$$

$$+ C\big(|| \operatorname{div} \beta||_{L^2(\Omega)} + || \operatorname{curl} \beta||_{L^2(\Omega)} + K||\beta||_{L^2(\Omega)}\big)^2. \quad (3.4.31)$$

Proof. Other than (3.4.26) for $p \neq 2$, all of the above inequalities are in Lemma 5.6 in [6]. To prove (3.4.26) for $p \neq 2$ we can argue in essentially the same way as the $p = 2$ case; by Stokes' theorem:

$$||\beta||^p_{L^p(\partial\Omega)} = \int_{\partial\Omega} \tilde{n}_i \tilde{n}^i |\beta|^p \, dS = \int_\Omega (D_i \tilde{n}^i)|\beta|^p + p\nabla\beta \cdot \beta|\beta|^{p-2}. \quad (3.4.32)$$

By Lemma 3.4.1, the first term is bounded by $K||\beta||^p_{L^p(\Omega)}$. To bound the second term, we just note that by Holder's inequality and Young's inequality, it is bounded by $||\nabla\beta||_{L^p(\Omega)}||\beta||^{p-1}_{L^p(\Omega)} \lesssim ||\nabla\beta||^p_{L^p(\Omega)} + ||\beta||^p_{L^p(\Omega)}$.

□

The estimates (3.4.25) will be used to show that the energy (defined in (3.5.8)) controls full derivatives of v. The estimates in Lemma 3.4.4 will be used to show that the energies control v on the boundary, and we will also use them with $\alpha = \nabla q$ for a function q to control solutions of the Dirichlet problem. We will assume in many of the following estimates that $K \leq 1$. This is only for notational convenience and is not essential to the arguments; many of the estimates will involve constants which can be bounded in terms of $1 + K$ and so this assumption allows us to ignore the unimportant dependence on K. We will make it clear when this assumption is used. Versions of these estimates with more explicit dependence on K can be found in [6].

First, we show that derivatives of q can be controlled by projected derivatives of q on the boundary and derivatives of Δq:

Proposition 3.4.1. *If $K \leq 1$ then for $r \geq 1$:*

$$||D^r q||_{L^2(\partial\Omega)} + ||D^r q||_{L^2(\Omega)} \leq C\Big(||\Pi D^r q||_{L^2(\partial\Omega)} + \sum_{s \leq r-1} ||D^s \Delta q||_{L^2(\Omega)} + ||Dq||_{L^2(\Omega)} \Big), \tag{3.4.33}$$

and for any $\delta > 0$:

$$||D^r q||_{L^2(\Omega)} + ||D^{r-1} q||_{L^2(\partial\Omega)} \leq \delta ||\Pi D^r q||_{L^2(\partial\Omega)}$$
$$+ C(1/\delta) \sum_{s \leq r-2} ||D^s \Delta q||_{L^2(\Omega)} + ||Dq||_{L^2(\Omega)}. \tag{3.4.34}$$

Proof. By (3.4.27) with $\beta = D^r q$:

$$||D^r q||^2_{L^2(\partial\Omega)} \leq ||\Pi D^r q||^2_{L^2(\partial\Omega)} + C\big(||D^{r-1}\Delta q||_{L^2(\Omega)} + K||D^r q||_{L^2(\Omega)} \big) ||D^r q||_{L^2(\Omega)}, \tag{3.4.35}$$

and by (3.4.30) with $\beta = D^{r-1} q$:

$$||D^r q||^2_{L^2(\Omega)} \leq C ||\Pi D^r q||_{L^2(\partial\Omega)} ||D^{r-1} q||_{L^2(\partial\Omega)}$$
$$+ C\big(||D^{r-1}\Delta q||_{L^2(\Omega)} + K ||D^{r-1} q||_{L^2(\partial\Omega)} \big)^2. \tag{3.4.36}$$

Combining these inequalities and using induction gives (3.4.33) and (3.4.34). □

We will use this proposition in two ways. First, our energy we will directly control $||\Pi D^r p||_{L^2(\Omega)}$ if the Taylor sign condition (3.1.7) holds and since $\Delta p = -(\partial_i v^j)(\partial_j v^i)$, we control the left-hand sides of (3.4.33)-(3.4.34) with $q = p$. We will also use this estimate to control derivatives of $D_t p$ on $\partial \mathcal{D}_t$, and we will rely on the observation that $\Pi D^r q$ is lower order if $q = 0$ on $\partial\Omega$. This is clear when $r = 0, 1$, and for $r = 2$ we have:

$$\Pi_i^j \Pi_k^\ell D_j D_\ell q = \Pi_i^j D_j \big(\Pi_k^\ell D_\ell q\big) - \Pi_i^j D_j (\Pi_k^\ell) D_\ell q, \tag{3.4.37}$$

and when $q = 0$ on $\partial\Omega$, the first term is zero and the second term is $-(\Pi_i^j D_j n_k) n^\ell D_\ell q$, so that $\Pi D^2 q = \theta D_n q$. We also record the $r = 3$ case for later use:

$$\Pi D^3 q = \overline{D}^{\,3} q - 2\theta \otimes (\theta \cdot \overline{D}\, q) + (\overline{D}\,\theta) D_N q + 3\theta \otimes (\overline{D}\, D_N q). \tag{3.4.38}$$

It will not be important in our argument exactly which indices appear where.

One can use the following heuristic argument from [6] to see what the higher-order version of the formula is. If $d(x) = \mathrm{dist}(x, \partial\Omega)$ then q/d is smooth up to the boundary, and:

$$\Pi D^r q = \Pi D^r \left(d \frac{q}{d} \right) = \sum_{s=0}^{r} \Pi(D^s d) \otimes D^{r-s} \left(\frac{q}{d} \right). \tag{3.4.39}$$

Restricting this formula to the boundary, we see that the $s = 0, 1$ terms drop out and that $q/d \sim n \cdot \partial q$. If the derivatives falling on d were purely tangential, then arguing as above we could replace $D^s d$ with $\overline{D}^{\,s-2}\theta$. We therefore write $D_i = (\Pi_i^j + n_i n^j) D_j$ and further note that $n^i n^j D_j D_k d = 0$ because d is the geodesic distance. Each time we make this subsitution, some derivatives will fall onto the factors of N we have introduced and this generates more factors of θ, but at the same time less derivatives land on the function q. This suggests that we should expect:

$$\Pi D^r q \sim \sum_{s=0}^{r-2} \overline{D}^{\,s} \theta \otimes D^{r-s} D_n q \tag{3.4.40}$$

Also note that the $s = r - 2$ term of the expansion (3.4.40) is $(\overline{D}^{\,r-2}\theta) D_n q$ and so if the lower order terms and $|Dq|^{-1}$ are bounded, this gives an estimate for θ in terms of q.

The rigorous version of these observations is:

Proposition 3.4.2. *Let $q : \mathcal{D}_t \to \mathbb{R}$ be a function. If $||\theta||_{L^\infty(\partial\Omega)} \leq 1$, then for $m = 0, 1$:*

$$||\Pi D^r q||^2_{L^2(\partial\Omega)} \leq ||\overline{D}^{\,r} q||_{L^2(\partial\Omega)} + 2||\overline{D}^{\,r-2}\theta||_{L^2(\partial\Omega)}\,||D_n q||_{L^\infty(\partial\Omega)}$$
$$+ C\Big(||\theta||_{L^\infty(\partial\Omega)} + \sum_{k \leq r-2-m} ||\overline{D}^{\,k}\theta||_{L^2(\partial\Omega)}\Big) \sum_{k\leq r-2+m} ||D^k q||_{L^2(\partial\Omega)}$$
$$+ C\sum_{k=1}^{r-1} ||D^{r-k} q||_{L^2(\partial\Omega)} \quad (3.4.41)$$

and if $|D_n q| > \delta_0 > 0$:

$$||\overline{D}^{\,r-2}\theta||_{L^2(\partial\Omega)} \leq C\delta_0^{-1}\Big(||\Pi D^r q||_{L^2(\partial\Omega)} + \sum_{k=1}^{r-1} ||D^{r-k} q||_{L^2(\partial\Omega)}\Big)$$
$$+ C\delta_0^{-1}\Big(||\theta||_{L^\infty(\partial\Omega)} + \sum_{k\leq r-3} ||\overline{D}^{\,r-3}\theta||_{L^2(\partial\Omega)}\Big) \sum_{k\leq r-1} ||D^k q||_{L^2(\partial\Omega)} \quad (3.4.42)$$

Combining these two propositions, we have:

Corollary 3.4.1. *If $K \leq 1$ and $q : \mathcal{D}_t \to \mathbb{R}$ is a function with $q = 0$ on $\partial\Omega$, then for $r \geq 3$:*

$$||D^{r-1} q||_{L^2(\partial\Omega)} \leq C\Big(||\overline{D}^{\,r-3}\theta||_{L^2(\partial\Omega)}||D_n q||_{L^\infty(\partial\Omega)} + ||D^{r-2}\Delta q||_{L^2(\Omega)}$$
$$+ C(||\theta||_{L^2(\partial\Omega)}, ..., ||\overline{D}^{\,r-4}\theta||_{L^2(\partial\Omega)})\Big(||D_n q||_{L^\infty(\partial\Omega)} + \sum_{s\leq r-3} ||D^s\Delta q||_{L^2(\Omega)} + ||Dq||_{L^2(\Omega)}\Big), \quad (3.4.43)$$

and for $r > 3$:

$$||D^{r-1} q||_{L^2(\partial\Omega)} + ||Dq||_{L^\infty(\partial\Omega)} \leq C||D^{r-2}\Delta q||_{L^2(\Omega)}$$
$$+ C(||\theta||_{L^2(\partial\Omega)}, ..., ||\overline{D}^{\,r-3}\theta||_{L^2(\partial\Omega)}) \sum_{s\leq r-3} ||D^s\Delta q||_{L^2(\Omega)}. \quad (3.4.44)$$

3.4.3 Estimates for v_ω

Unlike the previous section, in this section we will work on $\mathcal{D}_t$. We will therefore write:

$$\gamma_{ij} = \frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}\gamma_{ab}, \tag{3.4.45}$$

and similarly for γ_i^j, D_i, etc. In Section 3.D, we use the approach of [24] to show that $v_\omega = \operatorname{curl}\beta$, where β satisfies:

$$\Delta\beta = \omega \quad \text{in } \mathcal{D}_t, \tag{3.4.46}$$

$$\gamma_i^j\beta_j = 0, \quad j = 1,2,3, \text{ on } \partial\mathcal{D}_t, \tag{3.4.47}$$

$$D_n(\beta\cdot n) + \operatorname{tr}\theta\beta\cdot n = 0 \qquad \text{on } \partial\mathcal{D}_t. \tag{3.4.48}$$

Taking the divergence of (3.4.46) and noting that $D\cdot\beta|_{\partial\mathcal{D}_t} = \gamma D\cdot D(\gamma\cdot\beta) + D_n(\beta\cdot n) + \operatorname{tr}\theta\beta\cdot n = 0$, it follows that $\operatorname{div}\beta = 0$ in $\mathcal{D}_t$ if β. We have the basic elliptic estimate:

Lemma 3.4.5. *With β as defined above:*

$$||\beta||_{L^6(\mathcal{D}_t)} + ||D\beta||_{L^2(\mathcal{D}_t)} + ||\operatorname{curl}\beta||_{L^2(\mathcal{D}_t)} \lesssim ||\omega||_{L^{6/5}(\mathcal{D}_t)}. \tag{3.4.49}$$

Proof. First, by the Sobolev inequality (3.A.2), $||\beta||_{L^6(\mathcal{D}_t)} \lesssim ||D\beta||_{L^2(\mathcal{D}_t)}$. We next show that $||D\beta||_{L^2(\mathcal{D}_t)} \lesssim ||\operatorname{curl}\beta||_{L^2(\mathcal{D}_t)}$. Note that:

$$\int_{\mathcal{D}_t}\delta^{ij}\delta^{k\ell}D_i\beta_kD_j\beta_\ell = \int_{\mathcal{D}_t}\delta^{ij}\delta^{k\ell}D_i\beta_kD_\ell\beta_j + \int_{\mathcal{D}_t}\delta^{ij}\delta^{k\ell}D_i\beta_k\operatorname{curl}\beta_{j\ell}. \tag{3.4.50}$$

Integrating by parts, the first term is:

$$\int_{\mathcal{D}_t}\delta^{ij}\delta^{k\ell}D_i\beta_kD_\ell\beta_j = \int_{\partial\mathcal{D}_t}\delta^{ij}n^kD_i\beta_k\beta_j - \int_{\mathcal{D}_t}\delta^{ij}\delta^{k\ell}(D_\ell D_i\beta_k)\beta_j \tag{3.4.51}$$

The interior term vanishes since $\operatorname{div}\beta = 0$. To handle the boundary term, we note that since

$\gamma \cdot \beta = 0$ on $\partial\mathcal{D}_t$:

$$n^k \beta^i D_i \beta_k = n^k n^i (\beta^\ell n_\ell) D_i \beta_k = n^i (\beta^\ell n_\ell) D_n (n^k \beta_k) - n^i (\beta^\ell n_\ell) H n^k \beta_k$$

$$= \big(\operatorname{div} \beta - \gamma_j^i D_i(\gamma_\ell^k \beta^\ell)\big)(\beta^\ell n_\ell) = 0, \quad (3.4.52)$$

where we have used that $\operatorname{div} \beta = 0$. Returning to (3.4.50), we have:

$$||D\beta||^2_{L^2(\mathcal{D}_t)} \lesssim ||D\beta||_{L^2(\mathcal{D}_t)} || \operatorname{curl} \beta||_{L^2(\mathcal{D}_t)}, \quad (3.4.53)$$

which implies the bound for $||D\beta||_{L^2(\mathcal{D}_t)}$.

Finally we show that $|| \operatorname{curl} \beta||_{L^2(\mathcal{D}_t)} \lesssim ||\omega||_{L^{6/5}(\mathcal{D}_t)}$. Integrating by parts:

$$\int_{\mathcal{D}_t} |\operatorname{curl} \beta|^2 = \int_{\partial\mathcal{D}_t} (n \times \beta) \operatorname{curl} \beta - \int_{\mathcal{D}_t} \beta \operatorname{curl}^2 \beta. \quad (3.4.54)$$

Since the tangential components of β vanish on $\partial\mathcal{D}_t$, it follows that $n \times \beta = 0$. The interior term is bounded by $||\beta||_{L^6(\mathcal{D}_t)} ||\omega||_{L^{6/5}(\mathcal{D}_t)}$, which completes the proof. □

The above estimates combined with the elliptic estimates in the previous section will allow us to bound $||v_\omega||_{H^r(\mathcal{D}_t)}$. In the proof of the dispersive estimates, we will also need to bound $||V_\omega||_{L^p(\partial\mathcal{D}_t)}$ for $1 < p < 2$. Recall that in the interior, we have $V_\omega = \operatorname{curl} \beta$ with $\Delta\beta = \omega$. In the flat case ($h = 0$), a simple calculation using the Newton potential shows that for any $z \in \{(z_1, z_2, z_3) | z_3 \leq 0\}$, we have $|v_\omega(z)| = |\operatorname{curl} \beta(z)| \lesssim \frac{1}{1+|z|^2} ||(1 + |z|^2)\omega||_{L^1(\mathcal{D}_t)}$. Restricting this to $z = (x, 0) \in \partial\mathcal{D}_t$ gives that $V_\omega \in L^p(\partial\mathcal{D}_t)$ for $p > 1$. To handle the case with $h \neq 0$, in Proposition 3.D.1, we follow the approach of [25] to construct a Green's function for $\mathcal{D}_t$ which satisfies the same estimates as the Newtron potential, and this can be used to prove estimates for $||V_\omega||_{L^p(\partial\mathcal{D}_t)}$ for $1 < p$.

Proposition 3.4.3. *If* $||h||_{W^{4,\infty}(\mathbb{R}^2)} + ||h||_{W^{3,2}(\mathbb{R}^2)} \lesssim 1$*, then for* $2 \le p < \infty$, $0 \le r \le N_1 - 2$*:*

$$||\nabla^r V_\omega||_{L^p(\mathbb{R}^2)} + ||D^r v_\omega||_{L^2(\mathcal{D}_t)} \lesssim ||\omega||_{H^{N_1}_w(\mathcal{D}_t)}, \tag{3.4.55}$$

and for $1 < p \le 2$*:*

$$||V_\omega||_{L^p(\mathbb{R}^2)} \lesssim ||\omega||_{H^{N_1}_w(\mathcal{D}_t)} \tag{3.4.56}$$

If (3.2.9) holds, then by Hölder's inequality, we have $||h||_{W^{3,2}(\mathbb{R}^2)} \lesssim \varepsilon_0(1+t)^{-1/2}$ provided δ is sufficiently small so the assumption in the theorem holds. This condition is needed for a certain elliptic problem to be solvable; see Proposition 3.D.1.

Proof. First, by (3.3.5), $\nabla^r V_\omega = (D^r v_\omega)|_{\partial\mathcal{D}_t} - \nabla^r h (D_y v_\omega)|_{\partial\mathcal{D}_t} + (\nabla h)^r (D_y^r v_\omega)|_{\partial\mathcal{D}_t} + \dots$, up to similar terms. We show how to prove the estimates for the first term, as the other terms can be handled similarly. We consider the cases $1 < p < 2$ and $p \ge 2$ separately.

When $p \ge 2$, by Hölder's inequality and the Sobolev lemma, it suffices to control $||D^k V_\omega||_{L^2(\mathcal{D}_t)}$ for $0 \le k \le r+2$. Since $v_\omega \cdot n = 0$ on $\partial\mathcal{D}_t$, repeatedly applying the trace inequality (3.4.27) gives:

$$||D^k v_\omega||^2_{L^2(\partial\mathcal{D}_t)} \le C\Big(||D^k \omega||^2_{L^2(\mathcal{D}_t)} + (1+K)||v_\omega||^2_{L^2(\mathcal{D}_t)}\Big). \tag{3.4.57}$$

The constant here depends on bounds for $||\theta||_{L^\infty(\partial\mathcal{D}_t)}$ as well as $||\theta||_{H^{k-2}(\partial\mathcal{D}_t)}$ and by assumption these are both bounded. By the estimate (3.4.49), we have $||v_\omega||_{L^2(\mathcal{D}_t)} \lesssim ||\omega||_{L^{6/5}(\mathcal{D}_t)}$ and by Hölder's inequality, we have $||\omega||_{L^{6/5}(\mathcal{D}_t)} \lesssim ||\omega||_{H^{N_1}_w(\mathcal{D}_t)}$. Since $k \le r+2 \le N_1$ we bound the first term here as well.

The estimate (3.4.56) follows from (3.D.12). $\square$

3.5 Energy Estimates

The system (3.1.1)-(3.1.3) has a conserved energy:

$$E_0(t) = \frac{1}{2}\int_{\mathcal{D}_t} |v(t,x,y)|^2 dxdy + \frac{1}{2}\int_{\mathbb{R}^2} |h(t,x)|^2\, dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}\int_{-\infty}^{h(t,x)} |v(t,x,y)|^2\, dxdy + \frac{1}{2}\int_{\mathbb{R}^2} |h(t,x)|^2\, dx, \quad (3.5.1)$$

which follows from the following calculation:

$$\frac{d}{dt}E_0(t) = \int_{\mathcal{D}_t} v^i \partial_t v_i\, dxdy + \frac{1}{2}\int_{\mathbb{R}^2} \partial_t h |v|^2\, dx + \int_{\mathbb{R}^2} h\partial_t h\, dx \quad (3.5.2)$$

$$= -\int_{\mathcal{D}_t} v^i (v^k \partial_k v_i + \partial_i(p+y))\, dxdy + \frac{1}{2}\int_{\mathbb{R}^2} \partial_t h|v|^2\, dx + \int_{\mathbb{R}^2} h\partial_t h\, dx \quad (3.5.3)$$

$$= -\frac{1}{2}\int_{\partial\mathcal{D}_t} n_k v^k |v|^2\, dS - \int_{\partial\mathcal{D}_t} n_i v^i h\, dS + \frac{1}{2}\int_{\mathbb{R}^2} \partial_t h|v|^2\, dx + \int_{\mathbb{R}^2} h\partial_t h\, dx, \quad (3.5.4)$$

where we used that $\operatorname{div} v = 0$ in $\mathcal{D}_t$ and that $p = 0$ on $\partial\mathcal{D}_t$. Using (3.1.4) the first and third, and second and fourth terms here cancel.

To get higher-order energies, in the irrotational case ($\omega = 0$) one can use the system (3.3.1)-(3.3.2) directly to prove energy estimates. See [4] or [21] for this approach. In the case $\omega \neq 0$, the corresponding system (3.3.13)-(3.3.14) is more complicated to work with and we instead choose to model our approach on [6] and prove energy estimates for Euler's equation (3.1.1) -(3.1.3) directly. The advantage is that the estimates can be proved using elementary techniques, relying only on integration by parts and simple geometric facts (such as (3.4.40), (3.4.27)).

We define the projection γ as in (3.4.16). We will write:

$$\gamma^{ij} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\gamma^{ab}, \quad (3.5.5)$$

for γ expressed in the x-coordinates. We also write:

$$\gamma^{i_1\cdots i_r j_1\cdots j_r} = \gamma^{i_1 j_1}\cdots\gamma^{i_r j_r} \tag{3.5.6}$$

For $(0,r)-$tensors α,β, we define:

$$Q(\alpha,\beta) = \gamma^{i_1\cdots i_r j_1\cdots j_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r}. \tag{3.5.7}$$

The energies are:

$$\mathcal{E}^r(t) = \int_{\mathcal{D}_t}\delta^{ij}Q(D^r v_i, D^r v_j)\,dV + \int_{\partial\mathcal{D}_t}Q(D^r p, D^r p)|Dp|\,dS + \int_{\mathcal{D}_t}|D^{r-1}\omega|^2\,dV. \tag{3.5.8}$$

Since $p=0$ on $\partial\mathcal{D}_t$, by the estimate (3.5.41) (see the discussion after (3.4.37)), $Q(D^r p, D^r p) = Q(\overline{D}^{\,r-2}\theta,\overline{D}^{\,r-2}\theta)|D_n p|^2$ to highest order. In particular since $\theta\sim\nabla^2 h$, bounds for $\mathcal{E}^r$ imply bounds for h. Moreover, in Theorem 3.5.2, we will see that $||h||^2_{H^{N_0}(\mathbb{R}^2)} + ||\Lambda^{1/2}\varphi||^2_{H^{N_0-1}(\mathbb{R}^2)} \lesssim \mathcal{E}^{N_0}$ to highest order.

We will prove the energy estimates in the following sections assuming the following a priori bounds:

$$|\theta(t)| + \frac{1}{\iota_0(t)} \le K \qquad \text{on } \partial\mathcal{D}_t, \tag{3.5.9}$$

$$-n\cdot\partial p(t) \ge \delta_0 > 0 \qquad \text{on } \partial\mathcal{D}_t, \tag{3.5.10}$$

$$|D^2 p(t)| + |D_n D_t p(t)| \le L \qquad \text{on } \partial\mathcal{D}_t, \tag{3.5.11}$$

$$|Dv(t)| + |D^2 p(t)| \le M \qquad \text{on } \mathcal{D}_t. \tag{3.5.12}$$

Recall that we are writing $\iota_0(t)$ for the injectivity radius of $\partial\mathcal{D}_t$. We will assume in the estimates that $K\le 1$. This is only for notational convenience and is not essential to the arguments; many of the estimates will involve coefficients that can be bounded in terms of

$1+K$ and this allows us to ignore the unimportant dependence on K. We also remark that $\frac{1}{\iota_0} \leq ||\theta||_{L^\infty(\partial\mathcal{D}_t)}$ and so the definition of K is somewhat overcomplicated. We choose to keep track of both terms because it turns out that if one is interested in proving energy estimates which depend on as few derivatives of v as possible in L^∞, it is difficult to control the time evolution of ι_0. For this reason, in [6], the authors introduce another radius which they denote ι_1 (see Definition 3.5 there) which can be used to control ι_0. For our purposes this distinction will not be important, because we will eventually need to assume bounds for more derivatives of v in any case, but if one is interested in studying this problem with less regular data it may be useful to keep track of both terms.

The main result of this section is the following energy estimate:

Proposition 3.5.1. *Suppose that the a priori assumptions* (3.5.9)-(3.5.12) *hold. There are continuous functions $C_r = C_r(\delta_0^{-1})$ and homogeneous polynomials P_r with positive coefficients so that for $r \geq 0$:*

$$\Big|\frac{d}{dt}\mathcal{E}^r(t)\Big| \leq C_r(\delta_0^{-1})(K+L+M)\Big(\mathcal{E}^r(t) + (K+L+M)P_r(\mathcal{E}^*_{r-1}(t), K, L, M)\Big), \tag{3.5.13}$$

with $\mathcal{E}^*_{r-1} = \sum_{s\leq r-1}\mathcal{E}_s$.

We prove this in the next two subsections. Next, we relate the energy $\mathcal{E}^r$ and the a priori assumptions (3.5.9)-(3.5.12) to the dispersive variable u and the vorticity.

Lemma 3.5.1. *If the bootstrap assumptions* (3.2.9)-(3.2.11) *hold, then with:*

$$\mathcal{A}(t) = ||\theta(t)||_{L^\infty(\partial\mathcal{D}_t)} + \frac{1}{\iota_0(t)} + ||D^2 p||_{L^\infty(\partial\mathcal{D}_t)}$$
$$+ ||DD_t p||_{L^\infty(\partial\mathcal{D}_t)} + ||Dv(t)||_{L^\infty(\mathcal{D}_t)} + ||Dp(t)||_{L^\infty(\mathcal{D}_t)}, \tag{3.5.14}$$

and

$$\mathcal{B}(t) = ||h(t)||_{W^{4,\infty}(\mathbb{R}^2)} + ||\varphi(t)||_{W^{4,\infty}(\mathbb{R}^2)} + ||\omega(t)||_{H^{N_1}_w(\mathcal{D}_t)}, \tag{3.5.15}$$

we have:

$$\mathcal{A}(t) \lesssim \mathcal{B}(t)\left(1 + \mathcal{B}(t)\sqrt{\mathcal{E}_3^*(t)}\right), \tag{3.5.16}$$

where $\mathcal{E}_3^* = \sum_{s\leq 3} \mathcal{E}_s$*. Furthermore, if* $0 \leq t \leq T_{\varepsilon_0,\varepsilon_1}$ *with* $T_{\varepsilon_0,\varepsilon_1}$ *defined by* (3.1.15)*, then:*

$$-n\cdot\partial p(t) \geq \frac{1}{2}(-n\cdot\partial p(0)) \quad \text{on } \partial\mathcal{D}_t. \tag{3.5.17}$$

We remark that one could replace $||\omega||_{H_w^{N_1}(\mathcal{D}_t)}$ in (3.5.16) with an L^∞-based norm with fewer derivatives by using a Schauder estimate, but this will suffice for our purposes. We also note that the fact that $\sqrt{\mathcal{E}_3}$ shows up on the right-hand side of (3.5.16) is because we need to control $||DD_tp||_{L^\infty(\partial\mathcal{D}_t)}$. We bound this by Sobolev embedding and then the elliptic estimates in Section 3.4. Since $\Delta D_t p$ is cubic in the velocity (see (3.5.33)), this can be bounded by $\mathcal{B}^2\sqrt{\mathcal{E}_3^*}$.

Recall that $\varphi = \psi|_{\partial\mathcal{D}_t}$ where $n\cdot\partial\psi = n\cdot v$ on $\partial\mathcal{D}_t$. Since by Lemma 3.5.2, the energies control derivatives of v on $\partial\mathcal{D}_t$ as well as derivatives of θ, we have the following estimate, which is proved in Section 3.5.4.

Proposition 3.5.2. *With* φ_ω *defined by* (3.3.54)*, if* $||h||_{W^{4,\infty}(\mathbb{R}^2)} \ll 1$*, then for any* $r \geq 1$*:*

$$||h||^2_{H^r(\mathbb{R}^2)} + ||\Lambda^{1/2}\varphi_\omega||^2_{L^2(\mathbb{R}^2)} + ||\nabla\varphi_\omega||^2_{H^{r-1}(\mathbb{R}^2)} \leq C\mathcal{E}^r + \mathcal{A}P(\mathcal{E}_*^{r-1}, \mathcal{A}), \tag{3.5.18}$$

where $\mathcal{E}_*^r = \sum_{s\leq r}\mathcal{E}^s$ *and* $\mathcal{A}$ *defined by* (3.5.14)*.*

We will then see that the energy estimates (3.5.13) and this lemma imply:

Proposition 3.5.3. *If the bootstrap assumptions* (3.2.9)*-*(3.2.11) *hold, then:*

$$||\omega(t)||^2_{H_w^{N_1}(\mathcal{D}_t)} \leq ||\omega_0||^2_{H_w^{N_1}(\mathcal{D}_0)} + C_{N_1}\int_0^t \left(||u(s)||_{W^{N_1+2,\infty}(\mathbb{R}^2)} + ||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)}\right)||\omega(s)||^2_{H_w^{N_1}(\mathcal{D}_s)}\, ds. \tag{3.5.19}$$

We will need to take $N_1 \geq 6$ to prove the dispersive estimates and since our bootstrap assumptions only allow us to control $t||u(t)||_{W^{4,\infty}}$ uniformly in time, $||u||_{W^{N_1+2,\infty}(\mathbb{R}^2)}$ decays slightly slower than $1/t$. This is why we are only able to follow the solution until $T \sim \varepsilon_0^{-N}$.

Assuming these results for the moment, we can now provide the proofs of Theorem 3.2.4 and 3.2.3:

Proof of Theorem 3.2.4. The estimate (3.5.13) combined with (3.5.16) gives:

$$\begin{aligned}
&\mathcal{E}_{N_0}(t) - \mathcal{E}_{N_0}(0)\\
&\qquad \leq C^E_{N_0} \int_0^t \big(||u(s)||_{W^{4,\infty}(\mathbb{R}^2)} + ||\omega(s)||_{H^{N_1}_w(\mathcal{D}_s)}\big)\mathcal{E}_{N_0}(s)\, ds\\
&\qquad + C^E_{N_0} \int_0^t \big(||u(s)||_{W^{4,\infty}(\mathbb{R}^2)} + ||\omega(s)||_{H^{N_1}_w(\mathcal{D}_s)}\big)^2 \sqrt{\mathcal{E}^*_3(s)}\\
&\qquad\qquad \times P(\mathcal{E}_{N_0-1}(s), ||u||_{W^{4,\infty}(\mathbb{R}^2)}, ||\omega(s)||_{H^{N_1}_w(\mathcal{D}_s)})\, ds, \qquad (3.5.20)
\end{aligned}$$

for a constant $C^E_{N_0}$. With $M - 1/2$ the degree of the polynomial P in (3.5.20), the boostrap assumptions (3.2.9)-(3.2.11) show that if δ is sufficiently small then:

$$\mathcal{E}_{N_0}(t) \leq \mathcal{E}_{N_0}(0) + C^E_{N_0} \int_0^t \left(\frac{\varepsilon_0}{1+s} + \varepsilon_1\right)\varepsilon_0^2 (1+s)^{2\delta}\, ds \tag{3.5.21}$$

$$+ C^E_{N_0} \int_0^t \left(\frac{\varepsilon_0}{1+s} + \varepsilon_1\right)^2 \varepsilon_0^{2M}(1+s)^{2M\delta}\, ds \tag{3.5.22}$$

$$\leq \mathcal{E}_{N_0}(0) + C^E_{N_0}\left(\varepsilon_0^2(1+t)^{2\delta} + \varepsilon_1\varepsilon_0^2(1+t)^{1+2\delta} + \varepsilon_1^2\varepsilon_0^{2M}(1+t)^{1+2M\delta}\right), \tag{3.5.23}$$

as required. □

Proof of Theorem 3.2.3. By the interpolation inequality (3.A.3) combined with the estimate

(3.5.18) for $||u||_{H^{N_0}}$,

$$||u||_{W^{N_1+2,\infty}(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0}{(1+t)^{1-\sigma}}, \tag{3.5.24}$$

with $\sigma = \frac{N_1+2}{N_0-1}(1+\delta)$, provided the assumptions (3.2.9)-(3.2.11) hold. Recalling that $N_0 = 2NN_1$, this implies:

$$||u||_{W^{N_1,\infty}(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0}{(1+t)^{1-1/N}}. \tag{3.5.25}$$

Combining this with (3.5.19) and using the assumptions (3.2.9)-(3.2.11), we have:

$$||\omega(t)||^2_{H^{N_1}_w(\mathcal{D}_t)} \leq ||\omega(0)||^2_{H^{N_1}_w(\mathcal{D}_0)} + C_{N_1}\int_0^t \left(\varepsilon_0(1+s)^{-1+\sigma} + \varepsilon_1\right)\varepsilon_1^2\, ds \tag{3.5.26}$$

$$\leq \frac{1}{4}\varepsilon_1^2 + C_{N_1}\left(\varepsilon_0\varepsilon_1^2(1+t)^{\sigma} + \varepsilon_1^3(1+t)\right) \tag{3.5.27}$$

as required. □

As in [6], before proving the energy estimates (3.5.13), it is convenient to first prove that $\mathcal{E}^r$ controls norms of v, p and the second fundamental form θ.

3.5.1 Quantities controlled by $\mathcal{E}^r$

We start with the equations for ω and p. Taking the curl of (3.1.1) shows that ω satisfies:

$$D_t\omega_{ij} = \omega_{ik}D^k v_j. \tag{3.5.28}$$

Taking the divergence of (3.1.1) and using (3.1.2) gives that p satisfies:

$$\Delta p = -(D_i v^j)(D_j v^i) = -D_i(v^j D_j v^i), \tag{3.5.29}$$

where we used that $\operatorname{div} v = 0$. We will also need to use the equation for $D_t p$. We apply D_t to both sides of (3.5.29), and the right-hand side is:

$$-D_t(D_i(v^j D_j v^i)) = -D_i\left(D_t(v^j D_j v^i)\right) + D_k\left(v^k D_i(v^j D_j v^i)\right), \tag{3.5.30}$$

while:

$$D_t \Delta p = D_i D_t D^i p - D_i v^k D_k D^i p = \Delta D_t p - D_i\big(v^k D_k D^i p\big) - D_k(D_i v^k D^i p). \tag{3.5.31}$$

In particular, rearranging the indices this shows that:

$$\Delta D_t p = D_i \Big(v^k D_k D^i p - D_k v^i D^k p - D_t(v^j D_j v^i) + v^i D_k(v^j D_j v^k) \Big). \tag{3.5.32}$$

We shall need that the right-hand side of (3.5.32) is the divergence of a vector field, but for most of our applications it is more useful to use (3.1.2) and re-write this as:

$$\Delta D_t p = 4\,\mathrm{tr}\,\big((Dv)\cdot D^2 p\big) + 2\,\mathrm{tr}\,\big((Dv)^3\big) - (\Delta v)\cdot Dp, \tag{3.5.33}$$

where we are writing $((Dv)\cdot D^2 p)_{ij} = D_i v^k D_k D_j p$ and $((Dv)^3)_{ij} = D_i v^k D_k v^\ell D_\ell v_j$. The next lemma follows from these observations, the interpolation inequalities (3.A.6)-(3.A.7), and the fact that $[D_t, \partial_i] = -(\partial_i v^j)\partial_j$.

Lemma 3.5.2. *If $K \leq 1$ then there are constants $C_r > 0$ so that:*

$$||D_t D^r v + D^{r+1} p||_{L^2(\mathcal{D}_t)} + ||D_t D^{r-1}\omega||_{L^2(\mathcal{D}_t)} + ||\Delta D^{r-1} p||_{L^2(\mathcal{D}_t)}$$
$$\leq C_r ||Dv||_{L^\infty(\mathcal{D}_t)} \sum_{k=0}^{r} ||D^k v||_{L^2(\mathcal{D}_t)}, \tag{3.5.34}$$

$$||\Pi\big(D_t D^r p + (D^r v)\cdot Dp - D^r D_t p\big)||_{L^2(\partial\mathcal{D}_t)} \leq C_r \sum_{s=1}^{r-2} ||\Pi\big((D^{1+s} v)\cdot(D^{r-s} p)\big)||_{L^2(\partial\mathcal{D}_t)} \tag{3.5.35}$$

and

$$||D^{r-2}\Delta D_t p - (D^{r-2}\Delta v)\cdot Dp||_{L^2(\mathcal{D}_t)}$$

$$\leq C_r\big(||Dv||^2_{L^\infty(\mathcal{D}_t)} + ||Dp||_{L^\infty(\mathcal{D}_t)}\big)\Big(\sum_{s=1}^{r}||D^s v||^2_{L^2(\mathcal{D}_t)} + ||D^s p||_{L^2(\mathcal{D}_t)}\Big)$$

$$+ C_r||Dv||^2_{L^\infty(\mathcal{D}_t)}\sum_{s=1}^{r-1}||D^s v||_{L^2(\mathcal{D}_t)} \quad (3.5.36)$$

The elliptic estimates in Section 3.4 give us the following coercive estimates. These are essentially from [6]; the only difference here is that these estimates hold when $Vol\mathcal{D}_t = \infty$.

Lemma 3.5.3. *Suppose that $K \leq 1$. Then there are constants C_r with:*

$$||D^r v||^2_{L^2(\mathcal{D}_t)} \leq C_r\mathcal{E}^r, \tag{3.5.37}$$

$$||\Pi D^r p||^2_{L^2(\partial\mathcal{D}_t)} \leq ||Dp||_{L^\infty(\partial\mathcal{D}_t)}\mathcal{E}^r. \tag{3.5.38}$$

In addition, for $r \geq 1$:

$$||D^r p||^2_{L^2(\mathcal{D}_t)} + ||D^r p||^2_{L^2(\partial\mathcal{D}_t)} \leq C_r\big(||Dp||_{L^\infty(\partial\mathcal{D}_t)} + ||Dv||^2_{L^\infty(\mathcal{D}_t)}\big)^2\mathcal{E}^*_r, \tag{3.5.39}$$

with $\mathcal{E}^r_* = \sum_{k\leq r}\mathcal{E}_k$, and:

$$||\Pi D^r D_t p||^2_{L^2(\partial\mathcal{D}_t)} + ||D^{r-1}D_t p||^2_{L^2(\partial\mathcal{D}_t)} + ||D^r D_t p||^2_{L^2(\mathcal{D}_t)}$$

$$\leq C_r\big(||Dp||_{L^\infty(\mathcal{D}_t)} + ||Dv||^2_{L^\infty(\mathcal{D}_t)} + ||D_n D_t p||_{L^\infty(\partial\mathcal{D}_t)}||\theta||_{L^\infty(\mathcal{D}_t)}\big)\mathcal{E}^r(t)$$

$$+ P\Big(\mathcal{E}^*_{r-1}, ||Dp||_{L^\infty(\mathcal{D}_t)}, ||Dv||^2_{L^\infty(\mathcal{D}_t)}, ||D^2 p||_{L^\infty(\partial\mathcal{D}_t)}\Big). \quad (3.5.40)$$

Furthermore, if $-n \cdot \partial p \geq \delta_0 > 0$, then:

$$||\overline{D}^{r-2}\theta||^2_{L^2(\partial\mathcal{D}_t)} \leq ||(D_n p)^{-1}||_{L^\infty(\partial\mathcal{D}_t)}$$

$$\times \Big(\mathcal{E}^r + P(\mathcal{E}^*_{r-1}, ||Dv||_{L^\infty(\mathcal{D}_t)}, ||Dp||_{L^\infty(\mathcal{D}_t)}, ||D^2 p||_{L^\infty(\partial\mathcal{D}_t)})\Big) \quad (3.5.41)$$

where P is a homogeneous polynomial with positive coefficients.

Proof. The estimate (3.5.37) follows from (3.4.25) and (3.5.38) follows from the defintion of the boundary term in the energy. To prove (3.5.39), we apply (3.4.33), (3.5.34) and (3.5.38), which gives (3.5.39) with an extra term $||Dp||_{L^2(\mathcal{D}_t)}$ on the right-hand side. To control this, we integrate by parts twice and use (3.5.29):

$$\int_{\mathcal{D}_t} |Dp|^2 = -\int_{\mathcal{D}_t} p\Delta p = \int_{\mathcal{D}_t} pD_i(v^j D_j v^i) = \int_{\mathcal{D}_t} D_i p(v^j D_j v^i). \quad (3.5.42)$$

Bounding the right hand side by $||Dv||_{L^\infty(\mathcal{D}_t)}||Dp||_{L^2(\mathcal{D}_t)}||v||_{L^2(\mathcal{D}_t)}$ and dividing both sides by $||Dp||_{L^2(\mathcal{D}_t)}$ gives the result.

Similarly, applying (3.5.36), (3.4.33) and (3.4.41) gives (3.5.40) with an extra term $||DD_t p||_{L^2(\mathcal{D}_t)}$ on the right-hand side. This can be handled by using the fact that $D_t p = 0$ on $\partial\mathcal{D}_t$, the equation (3.5.32) and integrating by parts twice:

$$\int_{\mathcal{D}_t} |DD_t p|^2 = -\int_{\mathcal{D}_t} D_t p(D_i X^i) = \int_{\mathcal{D}_t} (D_i D_t p)X^i, \quad (3.5.43)$$

where $X_i = v^k D_k D_i p - D_k v_i D^k p - D_t(v^j D_j v_i) + v_i D_k(v^j D_j v^k)$. The result now follows after using (3.5.39) and (3.5.37) to control $||X||_{L^2(\mathcal{D}_t)}$.

The estimate (3.5.41) follows from (3.4.42) and the estimates we have just proved. □

3.5.2 Proof of Theorem 3.5.1

We start by applying Proposition 5.11 from [6] with $\alpha = -D^r p, \beta = D^{r-1}v$ and $\nu = |Dp|^{-1}$, which gives:

$$\frac{d}{dt}\mathcal{E}^r \le C\sqrt{\mathcal{E}^r}\big(||\Pi(D_tD^r p + (D_k p)D^r u^k)||_{L^2(\partial\mathcal{D}_t)} + ||D_tD^r u + D^{r+1}p||_{L^2(\mathcal{D}_t)}\big) + CK\mathcal{E}^r \tag{3.5.44}$$

$$+ C\big(||\,\mathrm{curl}\, D^{r-1}v||_{L^2(\mathcal{D}_t)} + ||\Delta D^{r-1}p||_{L^2(\mathcal{D}_t)} + K||D^{r-1}v||_{L^2(\mathcal{D}_t)} + ||D^r p||_{L^2(\mathcal{D}_t)}\big)^2. \tag{3.5.45}$$

By Lemma 3.5.3, every term except the first one above is bounded by the right-hand side of (3.5.13). By (3.5.35) and (3.5.40), it suffices to prove the following bound:

$$\sum_{s=1}^{r-2} ||\Pi((D^{1+s}u)\cdot(D^{r-s}p))||^2_{L^2(\partial\mathcal{D}_t)}$$

$$\le C(K+L+M)\Big(\mathcal{E}^r + (K+L+M)P(\mathcal{E}_0,..,\mathcal{E}_{r-1},K,L,M), \tag{3.5.46}$$

for a polynomial P. We write $(\Pi^{r-s}D^{r-s}p)_J = \gamma_J^I D_I^{r-s}$ and $(\Pi^{s+1}D^s v)_{Ji} = \gamma_J^I \gamma_j^i D_J v_j$. Then:

$$||\Pi\big((D^{s+1}v)\cdot(D^{r-s}p)\big)||_{L^2(\partial\mathcal{D}_t)} \le ||\,|\Pi^{s+1}D^s v|\,|\Pi^{r-s}D^{r-s}p|\,||_{L^2(\partial\mathcal{D}_t)}$$

$$+ ||\,|\Pi^s N^k D^s v_k|\,|\Pi^{r-s}N^k D^{r-s-1}D_k p|\,||_{L^2(\partial\mathcal{D}_t)} \tag{3.5.47}$$

We now apply the interplation inequality (3.A.6) which shows that see that each of these terms is bounded by a constant depending on K times (writing $L^p = L^p(\partial\mathcal{D}_t)$):

$$\Big(||D^2v||_{L^\infty} + \sum_{\ell=2}^{r-3}||D^\ell v||_{L^2}\Big)||\nabla^{r-1}p||_{L^2} + \Big(||D^3p||_{L^\infty} + \sum_{\ell=3}^{r-2}||D^\ell p||_{L^2}\Big)||\nabla^{r-2}v||_{L^2}$$

$$+ (||\theta||_{L^\infty} + ||\overline{D}^{\,r-3}\theta||_{L^2})\Big(||D^2v||_{L^\infty} + \sum_{\ell=2}^{r-3}||D^\ell v||_{L^2}\Big)\Big(||D^3v||_{L^\infty} + \sum_{\ell=3}^{r-2}||D^\ell p||_{L^2}\Big), \tag{3.5.48}$$

and using Lemma 3.5.2, this can be bounded by the right-hand side of (3.5.46).

3.5.3 Proof of Lemma 3.5.1

To control $||\theta||_{L^\infty(\partial\mathcal{D}_t)} + \frac{1}{\iota_0}$ we start by noting that $\frac{1}{\iota_0} \leq C||\theta||_{L^\infty(\partial\mathcal{D}_t)}$ and that by the elementary formula $\theta_{ij} = (1+|\nabla h|^2)^{-1/2}\nabla_i\nabla_j h$, we have $||\theta||_{L^\infty(\partial\mathcal{D}_t)} \leq C||h||_{C^2(\mathbb{R}^2)}$. We note that $\Delta|D^2\psi|^2 = |D^3\psi|^2 \geq 0$, so writing $v = D\psi + v_\omega$, applying the maximum principle to control $||D^2\psi||_{L^\infty(\mathcal{D}_t)} \leq ||D^2\psi||_{L^\infty(\partial\mathcal{D}_t)}$ and the estimate (3.4.55), we have:

$$||Dv||_{L^\infty(\mathcal{D}_t)} \leq ||D^2\psi||_{L^\infty(\mathcal{D}_t)} + ||Dv_\omega||_{L^\infty(\mathcal{D}_t)} \lesssim ||D^2\psi||_{L^\infty(\partial\mathcal{D}_t)} + ||\omega||_{H^{N_1}_w(\mathcal{D}_t)}. \tag{3.5.49}$$

To control $D^2\psi$ on $\partial\mathcal{D}_t$, we can either use (3.3.5) and (3.3.6) or just use the pointwise inequality (3.4.25) on $\partial\mathcal{D}_t$ which shows that $|D^2\psi| \lesssim |\Delta\psi| + |\Pi D^2\psi|$. By the projection formula (3.4.37) we have $|\Pi D^2\psi| \leq |\overline{D}^2\psi| + |\theta|(|D_N\psi| + |\overline{D}\,\psi|) \lesssim |\overline{D}^2\varphi| + |\theta|(|\mathcal{N}\varphi| + |\overline{D}\,\varphi|)$ where $\overline{D}$ denotes the covariant derivative on $\partial\mathcal{D}_t$. By the estimate for the Dirichlet-to-Neumann map (3.C.3), this proves the bound for $||Dv||_{L^\infty(\partial\mathcal{D}_t)}$.

The estimates for $||D^2p||_{L^\infty(\partial\mathcal{D}_t)}$ follow from the pointwise estimate (3.4.25), the fact that $\Delta p = -(Dv)\cdot(Dv)$ and the bounds we just proved. To bound $||DD_tp||_{L^\infty(\partial\mathcal{D}_t)}$, we apply Sobolev embedding (3.A.4) on $\partial\mathcal{D}_t$ and the elliptic estimate (3.4.34). It then suffices to control:

$$||\Pi D^3 D_t p||_{L^2(\partial\mathcal{D}_t)} + ||\Pi D^2 D_t p||_{L^2(\partial\mathcal{D}_t)} + \sum_{s\leq 2} ||D^s\Delta D_t p||_{L^2(\mathcal{D}_t)} + ||DD_tp||_{L^2(\mathcal{D}_t)}. \tag{3.5.50}$$

Using the identity (3.5.43) gives:

$$||DD_tp||_{L^2(\mathcal{D}_t)} \leq C\Big(||D^2p||_{L^\infty(\mathcal{D}_t)}||v||_{L^2(\mathcal{D}_t)} \\ + ||Dv||_{L^\infty(\mathcal{D}_t)}||Dp||_{L^2(\mathcal{D}_t)} + ||Dv||^2_{L^\infty(\mathcal{D}_t)}||Dv||_{L^2(\mathcal{D}_t)}\Big), \tag{3.5.51}$$

and using the estimates we have just proved and Lemma 3.5.2 gives that $||DD_t p||_{L^2(\mathcal{D}_t)}$ is bounded by the right-hand side of (3.5.16). To control $||\Pi D^3 D_t p||_{L^2(\partial\mathcal{D}_t)} + ||\Pi D^2 D_t p||_{L^2(\partial\mathcal{D}_t)}$, we use the formulas (3.4.37), (3.4.38) and the estimates we have just proved.

To get a lower bound for $\nabla_N p$ on $\partial\mathcal{D}_t$, we start by noting that since $p = 0$ on $\partial\mathcal{D}_t$ and $(D_t N^i)N_i = 0$, so that $D_t D_N p = D_N D_t p$ on $\partial\mathcal{D}_t$. Since $p = 0$ on $\partial\mathcal{D}_t$ and $(D_t N^i)N_i = 0$ it follows that $D_t \partial_N p = (D_t N^i)\partial_i p + \partial_N D_t p = \partial_N D_t p$. Applying Sobolev embedding on $\partial\mathcal{D}_t$, the estimate (3.5.40), and the bootstrap assumptions (3.2.9)-(3.2.11), we have:

$$|\nabla_N p(t)| \geq |\nabla_N p(0)| - \int_0^t |\nabla_N D_t p(s)|\, ds \geq |\nabla_N p(0)| - C\int_0^t \frac{\varepsilon_0^3}{(1+s)^2}(1+s)^\delta + \varepsilon_1^3, ds. \tag{3.5.52}$$

(Recall that $\Delta D_t p$ is cubic in the velocity.) The second term is bounded by $\frac{1}{2}|\nabla_N p(0)|$ so long as $t \leq C(|\nabla_N p(0)|^{-1})\varepsilon_1^{-1/3}$ and ε_0 is taken sufficiently small.

3.5.4 Proof of Proposition 3.5.2

We now show how the energies in the previous section control Sobolev norms of φ, h. Recall that $u = h + i\Lambda^{1/2}\varphi_\omega$, where $\varphi_\omega = \varphi + a_\omega$ and $\nabla a_\omega = V_\omega + \nabla h B_\omega$.

We begin by noting that by the definition of $\varphi_\omega = \varphi + a_\omega$, the fact that $\nabla a_\omega = V_\omega + \nabla h B_\omega$, and the fact that $B_\omega = -\nabla h \cdot V_\omega$ (since $v_\omega \cdot N = 0$) it suffices to prove the following estimate:

$$||h||^2_{H^r(\mathbb{R}^2)} + ||\Lambda^{1/2}\varphi||^2_{L^2(\mathbb{R}^2)} + ||\nabla\varphi||^2_{H^{r-1}(\mathbb{R}^2)} + ||\Lambda^{1/2}a_\omega||^2_{L^2(\mathbb{R}^2)} + ||V_\omega||^2_{H^{N-1}(\mathbb{R}^2)}$$

$$\lesssim \mathcal{E}^N + \mathcal{A}P(\mathcal{E}^{N-1}). \tag{3.5.53}$$

We start with bounds for h. By the elementary formula:

$$\theta_{ij} = \frac{1}{\sqrt{1+|\nabla h|^2}}\nabla_i\nabla_j h \tag{3.5.54}$$

we have $\nabla^r h \sim \nabla^{r-2}\theta + O(\nabla^{r-1}h, ..., \nabla h)$. We can therefore bound $||h||_{H^N(\mathbb{R}^2)}$ by the right-hand side of (3.5.18) provided we also control $||\nabla h||_{L^2(\mathbb{R}^2)} + ||h||_{L^2(\mathbb{R}^2)}$. Note that $||h||_{L^2(\mathbb{R}^2)} \leq E_0$ where E_0 is the conserved energy (defined in (3.5.1)), and a bound for $||\nabla h||_{L^2(\mathbb{R}^2)}$ follows from this and the bound for $||\nabla^2 h||_{L^2(\mathbb{R}^2)}$.

We now bound φ. First, we have:

$$-\int_{\partial\mathcal{D}_t} \varphi \mathcal{N} \varphi = -\int_{\partial\mathcal{D}_t} \psi D_N \psi = -\int_{\mathcal{D}_t} \psi \Delta \psi + \int_{\mathcal{D}_t} |\nabla \psi|^2 \leq ||v||^2_{L^2(\mathcal{D}_t)} + ||v_\omega||^2_{L^2(\mathcal{D}_t)}. \tag{3.5.55}$$

The left-hand side is:

$$||\mathcal{N}^{1/2}\varphi||^2_{L^2(\mathbb{R}^2)} \sim ||\Lambda^{1/2}\varphi||^2_{L^2(\mathbb{R}^2)}, \tag{3.5.56}$$

which follows from the remarks after Proposition 2.2 in [4].

To control $\Lambda^{1/2}a_\omega$, we note that by the fractional integration estimate (3.C.1), $||\Lambda^{1/2}a_\omega||_{L^2(\mathbb{R}^2)} = ||\Lambda^{-1/2}\Lambda a_\omega||_{L^2(\mathbb{R}^2)} \lesssim ||\Lambda a_\omega||_{L^{4/3}(\mathbb{R}^2)}$ and by the fact that the Riesz transform is bounded on $L^{4/3}$ it follows that $||\Lambda^{1/2}a_\omega||_{L^2(\mathbb{R}^2)} \lesssim ||\nabla a_\omega||_{L^{4/3}(\mathbb{R}^2)}$. Since $\nabla a_\omega = V_\omega + \nabla h B_\omega$, we have

$$||\Lambda^{1/2}a_\omega||_{L^2(\mathbb{R}^2)} \lesssim ||(1+|x|^2)^{1/2}V_\omega||_{L^2(\mathbb{R}^2)} + ||\nabla h||_{L^\infty(\mathbb{R}^2)}||(1+|x|^2)^{1/2}B_\omega||_{L^2(\mathbb{R}^2)} \tag{3.5.57}$$

and by (3.4.55), this is bounded by the right-hand side of (3.5.18).

To control the higher norms of φ and V_ω, we use the following:

Lemma 3.5.4. *Under the hypotheses of Proposition 3.5.2, we have:*

$$||\nabla^r\varphi||^2_{L^2(\mathbb{R}^2)} + ||\nabla^{r-1}V_\omega||^2_{L^2(\mathbb{R}^2)} \lesssim \mathcal{E}^r + \mathcal{A}P(\mathcal{E}^{r-1}_*), \tag{3.5.58}$$

where $\mathcal{E}^{r-1}_* = \sum_{s\leq r-1}\mathcal{E}^{r-1}$ *and* $\mathcal{A}$ *is defined by* (3.5.14).

Proof. The estimates for V_ω follow from (3.4.55). To bound $\nabla^r\varphi$, we start with the fact that:

$$||D\psi||_{L^2(\mathcal{D}_t)} \lesssim ||v||_{L^2(\mathcal{D}_t)} + ||v_\omega||_{L^2(\mathcal{D}_t)}, \tag{3.5.59}$$

By the chain rule, we have:

$$||\nabla \varphi||_{L^2(\mathbb{R}^2)} \leq ||D\psi||_{L^2(\partial \mathcal{D}_t)} + ||\nabla h D_y \psi||_{L^2(\partial \mathcal{D}_t)}. \tag{3.5.60}$$

Bounds for the second term will follow in a similar way to the bounds for the first term so we just show how to bound the first term. By the inequality (3.4.27):

$$||D\psi||^2_{L^2(\partial \mathcal{D}_t)} \leq ||D_N \psi||^2_{L^2(\mathcal{D}_t)} + ||\Delta \psi||^2_{L^2(\mathcal{D}_t)} + K||D\psi||^2_{L^2(\mathcal{D}_t)} \tag{3.5.61}$$

$$\leq ||v \cdot N||^2_{L^2(\partial \mathcal{D}_t)} + K||D\psi||^2_{L^2(\mathcal{D}_t)}, \tag{3.5.62}$$

and so the trace inequality (3.4.26) and the estimate (3.5.59) imply:

$$||D\varphi||_{L^2(\mathbb{R}^2)} \lesssim ||Dv||_{L^2(\mathcal{D}_t)} + ||v||_{L^2(\mathcal{D}_t)} + ||v_\omega||_{L^2(\mathcal{D}_t)}, \tag{3.5.63}$$

where the implicit constant depends only on K. The first two terms are bounded by $\mathcal{E}^1 + \mathcal{E}^0$ and the last term can be bounded by $||\omega||_{H^{N_1}_w(\mathcal{D}_t)}$ by (3.4.3). To explain the strategy for higher-order derivatives we first consider what happens when $r = 2$. Using (3.4.27) again:

$$||D^2\psi||_{L^2(\partial \mathcal{D}_t)} \lesssim ||D_N D\psi||_{L^2(\partial \mathcal{D}_t)} + K||D\psi||_{L^2(\mathcal{D}_t)}. \tag{3.5.64}$$

By the estimate (3.4.27):

$$||D_N D\psi||_{L^2(\partial \mathcal{D}_t)} \lesssim ||\Pi D_N D\psi||_{L^2(\partial \mathcal{D}_t)}$$

$$+ ||\operatorname{div} D_N D\psi||_{L^2(\mathcal{D}_t)} + ||\operatorname{curl} D_N D\psi||_{L^2(\mathcal{D}_t)} + K||D_N \psi||_{L^2(\mathcal{D}_t)}. \tag{3.5.65}$$

Note that:

$$\Pi^i_j D_N D_i \psi = \Pi^i_j D_i D_N \psi - (\Pi^i_j D_i N^k) D_k \psi. \tag{3.5.66}$$

The first term is $\overline{D}\,(v \cdot N)$ and the second term is $-\theta^k_i D_k \psi$. Also both div $D_N D\psi$ and curl $D_N \psi$

are lower order. The first is because to highest order it is $D_N \Delta \psi = 0$ and the second because curl $D\psi = 0$.

Therefore we have:

$$||D^2\psi||_{L^2(\partial\mathcal{D}_t)} \lesssim ||\overline{D}\,(v \cdot N)||_{L^2(\partial\mathcal{D}_t)} + K||D\psi||_{L^2(\partial\mathcal{D}_t)} + ||D\psi||_{L^2(\mathcal{D}_t)}. \tag{3.5.67}$$

Using the trace inequality to bound the first term and the above argument to bound the lower-order norms of ψ gives that:

$$||D^2\psi||_{L^2(\partial\mathcal{D}_t)} \lesssim ||D^2 v||_{L^2(\mathcal{D}_t)}||Dv||_{L^2(\mathcal{D}_t)} + ||v||_{L^2(\mathcal{D}_t)} + ||v_\omega||_{L^2(\mathcal{D}_t)}, \tag{3.5.68}$$

where the implicit constant depends only on K.

We now prove a higher-order version of this. Repeatedly applying the chain rule (3.3.5), to highest order we have:

$$\nabla^r \varphi \sim \nabla^r \psi + \nabla^r h(D_y \psi) + ... \tag{3.5.69}$$

where the missing terms are all bounded pointwise by $\sum_{k \le r-1} |D^k_{x,y}\psi|$ times a polynomial in $\sum_{k\le r-1} |\nabla^k h|$. We now want to replace $\nabla^r\psi$ with $\nabla^{r-1}\nabla_N\psi \sim \overline{D}^{\,r-1}(v \cdot N)$ and lower order terms. By the inequality (3.4.27):

$$||\nabla^r\psi||_{L^2(\partial\mathcal{D}_t)} \lesssim ||\nabla_N \nabla^{r-1}\psi||_{L^2(\partial\mathcal{D}_t)} + ||\nabla^{r-1}\psi||_{L^2(\mathcal{D}_t)}, \tag{3.5.70}$$

with implicit constant depending on K. Next, with $\beta = \nabla^{r-1}\psi$, we apply the estimate (3.4.27) and have:

$$\begin{aligned} ||(n\cdot\partial)\nabla^{r-1}\psi||_{L^2(\partial\mathcal{D}_t)} &\lesssim ||\Pi(n\cdot\partial)\nabla^{r-1}\psi||_{L^2(\partial\mathcal{D}_t)} \\ &+ ||\operatorname{div} n \cdot \partial\nabla^{r-2}\psi||_{L^2(\mathcal{D}_t)} + ||\operatorname{curl} n \cdot \partial\nabla^{r-2}\psi||_{L^2(\mathcal{D}_t)} + ||n\cdot\partial\nabla^{r-2}\psi||_{L^2(\mathcal{D}_t)}. \end{aligned} \tag{3.5.71}$$

The interior terms are all lower order by the same observation as above, and so we just need to deal with the boundary term. We note that:

$$\Pi_J^I (n \cdot \partial) \nabla_I^{r-1} \psi = \Pi_J^I \nabla_I^{r-1} (n \cdot \partial) \psi - \sum_{K,L} \Pi_J^I (\partial_K^k n)(\partial_L^{r-s} \nabla \psi) \tag{3.5.72}$$

where the sum is over all multi-indices K, L with $K + L = I$ and $|K| \leq |I| - 1$.

Since $n \cdot \partial\psi = n \cdot v$ on $\partial\mathcal{D}_t$, using (3.4.2) to replace $\Pi_J^I \nabla_I^{r-1} n \cdot \partial\psi$ with $\overline{D}^{r-1} n \cdot \partial\psi$ and applying Lemma 3.5.2 to control $\overline{D}^{r-1}(n \cdot v)$ by the energy shows that the first term in (3.5.72) is controlled by the energy. The worst term appearing in the sum in (3.5.72) from the point of view of the regularity of θ is the case $K = I$. This involves $r - 1$ projected derivatives of n and by Proposition 4.11 of [6] and the definition $\theta = \Pi\nabla N$, this can be bounded by $||\overline{D}^{r-2}\theta||_{L^2(\partial\mathcal{D}_t)}$ to highest order. We can now use induction and interpolation (3.A.6) to deal with the lower-order terms.

Having now bounded φ, let us see how to control V_ω and B_ω. First, since $v_\omega \cdot n = 0$ on $\partial\mathcal{D}_t$, we have $B_\omega = V_\omega \cdot \nabla h$ and so it is enough to bound V_ω. Since $V_\omega = v_\omega|_{\partial\mathcal{D}_t} = (v - \nabla\psi)|_{\partial\mathcal{D}_t}$, estimates for V_ω follow from the above estimates for ψ and the estimates in Lemma 3.5.3.

□

3.5.5 Proof of Proposition 3.5.3

A short calculation using the fact that $[D_t, D] = -Dv^k D_k$, $D_t(1 + |z|^2)^2 = 4|z|^2 z \cdot v$ and the equation for the vorticity (3.5.28) shows that:

$$D_t D^m ((1 + |z|^2)^2 \omega) = (1 + |z|^2)^2 \left(D^{m+1} v \cdot \omega + Dv \cdot D^m \omega \right) + R, \tag{3.5.73}$$

where R is a sum of terms which can be bounded pointwise by

$$(1 + |z|^2)^2 \sum_{a=1}^{m} |D^a v(z)||D^{m-a} \omega(z)|.$$

We next write $v = D\psi + v_\omega$ and the result as:

$$D_t((1+|z|^2)^2 D^m\omega) = (1+|z|^2)^2\Big(D^{m+2}\psi\cdot\omega + D^2\psi\cdot D^m\omega$$

$$+ D^{m+1}v_\omega\cdot\omega + Dv_\omega\cdot D^m\omega\Big) + R. \quad (3.5.74)$$

Taking $m \leq N_0$ By the Reynolds transport theorem, the above calculation and Sobolev embedding, we have:

$$\frac{d}{dt}||D^m\omega(t)||^2_{L^2_w} \quad (3.5.75)$$

$$\lesssim \int_{\mathcal{D}_t}(1+|z|^2)^2\big(|D^{m+2}\psi||\omega| + |D^2\psi||D^m\omega| \quad (3.5.76)$$

$$+ |D^{m+1}v_\omega||\omega| + |Dv_\omega||D^m\omega| + R\big)|D^m\omega|\,dz \quad (3.5.77)$$

$$\lesssim \Big(||D^2\psi||_{W^{m,\infty}(\mathcal{D}_t)} + ||D^{m+1}v_\omega||_{L^2(\mathcal{D}_t)} + ||v_\omega||_{L^\infty(\mathcal{D}_t)}\Big)||\omega||^2_{H^{N_1}_w(\mathcal{D}_t)} \quad (3.5.78)$$

To control the first term, we use the maximum principle as in the proof of Lemma 3.5.1, which gives that $||D^2\psi||_{W^{m,\infty}(\mathcal{D}_t)} \leq ||D^2\psi||_{W^{m,\infty}(\partial\mathcal{D}_t)}$. Using the chain rule and (3.3.6) repeatedly shows that $||D^s\psi||_{L^\infty(\partial\mathcal{D}_t)} \lesssim ||D\varphi||_{W^{s-1,\infty}(\mathbb{R}^2)}(1+||h||_{W^{s,\infty}(\mathbb{R}^2)}) \lesssim ||u||_{W^{s+1,\infty}(\mathbb{R}^2)}$,

To control the other two terms from (3.5.78), we use (3.4.55):

$$||v_\omega||_{L^\infty(\mathcal{D}_t)} + ||D^{m+1}v_\omega||_{L^2(\mathcal{D}_t)} \lesssim ||\omega||_{H^{N_1}_w(\mathcal{D}_t)}, \quad (3.5.79)$$

which proves (3.5.19).

We also note the following, which is used in the proof of Corollary 3.1.1:

Lemma 3.5.5. *If $\omega_0|_{\partial\mathcal{D}_0} = 0$ and $\int_0^T ||\partial v||_{L^\infty(\partial\mathcal{D}_s)} < \infty$, for some $T > 0$, then $\omega|_{\partial\mathcal{D}_t} = 0$ for $t \leq T$.*

Proof. Changing to Lagrangian coordinates and letting μ_γ denote the volume element on $\partial\Omega$

with respect to the metric γ, we have:

$$\frac{d}{dt}\int_{\partial\mathcal{D}_t}|\omega(t)|^2\,dS=\frac{d}{dt}\int_\Omega|\omega(t)|^2d\mu_\gamma=\int_\Omega D_t\omega(t)\cdot\omega(t)d\mu_\gamma+\int_\Omega|\omega(t)|^2D_td\mu_\gamma. \quad (3.5.80)$$

By Lemma 3.9 in [6], we have $D_td\mu_\gamma=(\operatorname{tr}h-h_{nn})d\mu_\gamma$ where $h=\frac{1}{2}D_tg$ with g the metric in Lagrangian coordinates (defined in (3.4.4)) and $h_{nn}=h(n,n)|_{\partial\mathcal{D}_t}$. A simple calculation using (3.4.4) and the fact that $D_t\frac{d}{dy}x^i=\frac{d}{dy}V^i$ gives that $|D_td\mu_\gamma|\le C||\partial v||_{L^\infty(\partial\Omega)}d\mu_\gamma$, so by (3.5.28), (3.5.80) gives:

$$\frac{d}{dt}||\omega(t)||^2_{L^2(\partial\mathcal{D}_t)}\le C||\partial v||_{L^\infty(\partial\mathcal{D}_t)}||\omega(t)||^2_{L^2(\partial\mathcal{D}_t)}. \quad (3.5.81)$$

Multiplying both sides by the integrating factor $e^{-C\int_0^t||\partial v(s)||_{L^\infty(\partial\mathcal{D}_s)}\,ds}$ and integrating gives that:

$$||\omega(t)||^2_{L^2(\partial\mathcal{D}_t)}\le C\exp\Big(\int_0^t||\partial v(s)||_{L^\infty(\partial\mathcal{D}_s)}\Big)||\omega(0)||^2_{L^2\partial\mathcal{D}_0)}, \quad (3.5.82)$$

from which the result follows. $\square$

3.6 Estimates for terms involving the vorticity

In this section, we prove estimates for the terms $g_2,...,g_5$ from (3.3.76). We recall that R_j denotes the Riesz transform and Λ^s denotes fractional differentiation on $\mathbb{R}^2$. We will also ignore the difference between Reu, Imu and just write u. Then the terms we want to estimate are:

$$g_2(t)=\int_0^t e^{is\Lambda^{1/2}}R\cdot V_\omega(s)\,ds, \quad (3.6.1)$$

$$g_3(t)=\int_0^t e^{is\Lambda^{1/2}}\Lambda^{1/2}\Big(\big((R\cdot V_\omega)(\Lambda^{1/2}u)\big)-\nabla\cdot\big(uV_\omega\big)+\Lambda(uR\cdot V_\omega)\Big)\,ds, \quad (3.6.2)$$

$$g_4(t)=\int_0^t e^{is\Lambda^{1/2}}\Lambda^{1/2}(R\cdot V_\omega)^2\,ds. \quad (3.6.3)$$

In the next three sections, we prove:

Proposition 3.6.1. *If* (3.2.9) *holds with* $\varepsilon_0 \ll 1$*, then for* $I = 2, 3, 4$*:*

$$||\nabla^k e^{-it\Lambda^{1/2}} g_I||_{L^\infty(\mathbb{R}^2)} \lesssim \frac{\varepsilon_0^2}{1+t} + \varepsilon_1(1+t), \qquad 0 \leq k \leq 4, \tag{3.6.4}$$

$$||\Lambda^\iota x g_I||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_0^2(1+t)^\delta + \varepsilon_1(1+t)^2. \tag{3.6.5}$$

In the below estimates, we will see that the term that contributes the fastest-growing term is g_2. This is because g_2 depends on the vorticity linearly. The term g_3 satisfies better estimates than (3.6.4)-(3.6.5) because it involves a factor of u, which we expect to decay, while the term g_4 satisfies a slightly better estimate than g_2 because it is quadratic in the vorticity ω.

3.6.1 Estimates for g_2

Lemma 3.6.1. *If* v *satisfies* (3.2.9),(3.2.10) *with* $\varepsilon_0 \ll 1$*, then:*

$$||\nabla^k e^{-it\Lambda^{1/2}} g_2(t)||_{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon_1(1+t) \qquad 0 \leq k \leq 4, \tag{3.6.6}$$

$$||\Lambda^\iota x g_2(t)||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_1(1+t)^2. \tag{3.6.7}$$

Proof. The estimates (3.6.6) are a direct consequence of the following estimates:

$$||\nabla^k e^{-it\Lambda^{1/2}} g_2(t)||_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t ||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)}\, ds, \tag{3.6.8}$$

$$||\Lambda^\iota x g_2(t)||_{L^2(\mathbb{R}^2)} \lesssim \int_0^t (1+s)||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)}\, ds, \tag{3.6.9}$$

and the bootstrap assumption (3.2.11).

To prove (3.6.8), we use Sobolev embedding:

$$||\nabla^k e^{i(t-s)\Lambda^{1/2}} R \cdot V_\omega(s)||_{L^\infty(\mathbb{R}^2)} \lesssim ||R \cdot V_\omega(s)||_{H^{2+k}(\mathbb{R}^2)} \lesssim ||V_\omega(s)||_{H^{2+k}(\mathbb{R}^2)}, \tag{3.6.10}$$

and then the estimate (3.4.3). (A better estimate is possible since we have not made use of the dispersive estimate (3.C.7), but this estimate will suffice for our purposes.)

To prove (3.6.9), we note that by Plancherel's theorem, $||\Lambda^\iota x e^{it\Lambda^{1/2}} g_2||_{L^2} = |||\xi|^\iota \partial_\xi (e^{it|\xi|^{1/2}} \hat{g}_2)||_{L^2}$. We have:

$$\partial_\xi e^{it|\xi|^{1/2}} \hat{g}_2 = \partial_\xi \left(\int_0^t e^{is|\xi|^{1/2}} \frac{\xi}{|\xi|} \cdot \hat{V}_\omega(s,\xi)\, ds \right) \tag{3.6.11}$$

$$= \int_0^t s \frac{\xi}{|\xi|^{3/2}} e^{is|\xi|^{1/2}} \hat{V}_\omega(s,\xi)\, ds + \int_0^t e^{is|\xi|^{1/2}} \partial_\xi \left(\frac{\xi}{|\xi|} \right) \hat{V}_\omega(s,\xi)\, ds \tag{3.6.12}$$

$$+ \int_0^t e^{is|\xi|^{1/2}} \partial_\xi \hat{V}_\omega(s,\xi)\, ds \tag{3.6.13}$$

$$\equiv \int_0^t s\hat{g}_2^1(s,\xi)\, ds + \int_0^t \hat{g}_2^2(s,\xi)\, ds + \int_0^t \hat{g}_2^3(s,\xi)\, ds. \tag{3.6.14}$$

With $p_1 = 2(2-\iota)/3$, by the fractional integration lemma (3.C.1), we have:

$$||\Lambda^\iota g_2^1||_{L^2} \lesssim ||\Lambda^{-1/2+\iota} V_\omega||_{L^2} \lesssim ||V_\omega||_{L^{p_1}}, \tag{3.6.15}$$

Since $p_1 > 1$, by (3.4.3) we therefore have:

$$\int_0^t s||\Lambda^\iota g_2^1(s)||_{L^2(\mathbb{R}^2)} \lesssim \int_0^t s||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)}\, ds. \tag{3.6.16}$$

Writing $m_0(\xi) = \partial_\xi(|\xi|^{-1}\xi)$, we see that $m_0(\nabla)\Lambda$ is an operator of order 0. Taking $p_2 = 2/(2-\iota) > 1$, applying fractional integration (3.C.1) and e.g. the Hörmander-Mikhlin multiplier theorem along with (3.4.3) gives:

$$\begin{aligned} ||\Lambda^\iota g_2^2||_{L^2} &\lesssim \int_0^t ||(\Lambda m_0(\nabla))\Lambda^{-1+\iota} V_\omega(s)||_{L^2} \\ &\lesssim \int_0^t ||\Lambda^{-1+\iota} V_\omega(s)||_{L^2}\, ds \lesssim \int_0^t ||V_\omega(s)||_{L^{p_2}}\, ds \lesssim \int_0^t ||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)}\, ds, \end{aligned} \tag{3.6.17}$$

by (3.4.56).

To control $\Lambda^\iota g_2^3$, we write $\Lambda^\iota = \Lambda^{\iota-1}\Lambda = -\Lambda^{\iota-1}R\cdot\nabla$, where R denotes the Riesz transform. Using fractional integration (3.C.1) again, we have:

$$||\Lambda^\iota(xV_\omega)||_{L^2} = ||\Lambda^{-1+\iota}R\cdot\nabla(xV_\omega)||_{L^2} \lesssim ||\nabla(xV_\omega)||_{L^{p_1}}, \tag{3.6.18}$$

again with $p_1 = 2(2-\iota)/3$. Combining this with (3.4.3) gives (3.6.9). $\square$

3.6.2 Estimates for g_3

We now bound the term involving both u and V_ω. This is:

$$g_3(t) = \int_0^t e^{is\Lambda^{1/2}} N_1(u,w)\,ds = \sum_{I=1,2,3}\int_0^t e^{is\Lambda^{1/2}} G_3^I(s)\,ds, \tag{3.6.19}$$

with:

$$G_3^1 = \Lambda^{1/2}\big((R\cdot V_\omega)(\Lambda^{1/2}u)\big), \quad G_3^2 = -\nabla\cdot(V_\omega u), \quad G_3^3 = \Lambda((R\cdot V_\omega)u). \tag{3.6.20}$$

Lemma 3.6.2. *If (3.2.9)-(3.2.11) hold with $\varepsilon_0 \ll 1$, then:*

$$||\nabla^k e^{-it\Lambda^{1/2}} g_3(t)||_{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon_0\varepsilon_1(1+t)^{1/N} \quad \text{if } k \le N_1 - 3, \tag{3.6.21}$$

$$||\Lambda^\iota x g_3(t)||_{L^2} \lesssim \varepsilon_0\varepsilon_1(1+t)^{1+\delta/2}. \tag{3.6.22}$$

Proof. The estimates for each of the terms G_3^I, $I = 1,2,3$ are similar, so we just show how to bound G_3^1. The estimates (3.6.21) and (3.6.22) will follow from:

$$||\nabla^k e^{-it\Lambda^{1/2}} G_3^1(t)||_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t ||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)} ||u(s)||_{W^{k+3,\infty}(\mathbb{R}^2)}\,ds \tag{3.6.23}$$

$$||\Lambda^\iota(xG_3^1(t))||_{L^2} \lesssim \int_0^t (1+s)||\omega(s)||_{H_w^{N_1}(\mathcal{D}_s)}\Big(||u(s)||_{W^{4,\infty}} + ||u(s)||_{H^{N_0}}\Big)\,ds. \tag{3.6.24}$$

Assuming these hold for the moment, using the interpolation inequality (3.A.8) and the bootstrap assumptions (3.2.9)-(3.2.11), we have:

$$||\omega(s)||_{H^{N_1}_w(\mathcal{D}_s)}||u(s)||_{W^{k+3,\infty}(\mathbb{R}^2)} \lesssim \varepsilon_1\varepsilon_0(1+s)^{-1+1/N}, \tag{3.6.25}$$

which implies (3.6.21). The estimate (3.6.22) follows directly from (3.6.24) and (3.2.9)-(3.2.11).

We now prove (3.6.23). Applying Sobolev embedding and using the fact that the Riesz transform is bounded on L^2, we have:

$$\begin{aligned}||e^{i(t-s)\Lambda^{1/2}}\nabla^k\Lambda^{1/2}(R\cdot V_\omega(s)\Lambda^{1/2}u(s))||_{L^\infty} &\lesssim ||(R\cdot V_\omega(s))(\Lambda^{1/2}u(s))||_{H^{k+2}}\\ &\lesssim ||V_\omega(s)||_{H^{k+2}}||u(s)||_{W^{k+3,\infty}}. \end{aligned}\tag{3.6.26}$$

Therefore, using (3.4.3), the interpolation estimate (3.A.8) and the bootstrap assumptions (3.2.9)-(3.2.10):

$$\begin{aligned}||\nabla^k e^{-it\Lambda^{1/2}}G_3^1||_{L^\infty(\mathbb{R}^2)} &\lesssim \int_0^t ||e^{i(t-s)\Lambda^{1/2}}\nabla^k\Lambda^{1/2}(R\cdot V_\omega(s)\Lambda^{1/2}u(s))||_{L^\infty}\,ds\\ &\lesssim \int_0^t \varepsilon_0\varepsilon_1(1+s)^{\delta}(1+s)^{-1+\sigma}\,ds \lesssim \varepsilon_0\varepsilon_1(1+t)^{\delta+\sigma} \lesssim \frac{\varepsilon_0^2}{1+t}+\varepsilon_1^2(1+t)^{1+\delta+\sigma}, \end{aligned}\tag{3.6.27}$$

where $\sigma = \frac{k+3}{N_0}(1+\delta)$. Provided $\frac{1}{N_0}\sim\delta$, $\sigma\le 2\delta$ and this gives (3.6.21). Again, using the dispersive estimate (3.C.7) would lead to a better estimate but this one will suffice.

To prove the bound for xG_3^1, we write:

$$\partial_\xi \hat{G}_3^1 = \int_{\mathbb{R}^2} \partial_\xi \left(e^{is|\xi|^{1/2}} |\xi|^{1/2} \frac{(\xi-\eta)_\ell}{|\xi-\eta|} |\eta|^{1/2} \hat{V}_\omega^\ell(\xi-\eta)\hat{u}(\eta) \right) d\eta \tag{3.6.28}$$

$$= \int_{\mathbb{R}^2} e^{is|\xi|^{1/2}} |\eta|^{1/2} \left(is \frac{(\xi-\eta)_\ell}{|\xi-\eta|} + |\xi|^{1/2} m_\ell(\xi-\eta) \right. \tag{3.6.29}$$

$$\left. + \frac{1}{2} \frac{\xi}{|\xi|^{3/2}} \frac{(\xi-\eta)_\ell}{|\xi-\eta|} \right) \hat{V}_\omega^\ell(\xi-\eta)\hat{u}(\eta) \Big) \, d\eta \tag{3.6.30}$$

$$+ \int_{\mathbb{R}^2} e^{is|\xi|^{1/2}} |\xi|^{1/2} \frac{(\xi-\eta)_\ell}{|\xi-\eta|} |\eta|^{1/2} \partial_\xi \hat{V}_\omega^\ell(\xi-\eta)\hat{u}(\eta)\, d\eta, \tag{3.6.31}$$

with $m_\ell(\xi-\eta) = \partial_\xi \left(\frac{(\xi-\eta)_\ell}{|\xi-\eta|} \right)$ and $m_1(\xi) = \partial_\xi |\xi|^{1/2}$.

We fix a small parameter $\epsilon > 0$. After applying Λ^ι the L^2 norm of the first term in (3.6.30) is bounded by:

$$\int_0^t s||e^{is\Lambda^{1/2}}\Lambda^\iota\big((R\cdot V_\omega)(\Lambda^{1/2}u)\big)||_2\, ds \lesssim \int_0^t s||V_\omega(s)||_{W^{1,2/(1-2\epsilon)}}||u(s)||_{W^{1,1/\epsilon}}\, ds$$

$$\lesssim \int_0^t s||V_\omega(s)||_{W^{1,2/(1-2\epsilon)}}||u(s)||_\infty^{1-2\epsilon}||u(s)||_2^{2\epsilon}\, ds \tag{3.6.32}$$

where we have used Hölder's inequality, the fractional product rule (3.C.2) and the boundedness of the Riesz transform. Using (3.4.3) and the bootstrap assumptions (3.2.9)-(3.2.11), we have:

$$\int_0^t s||e^{is\Lambda^{1/2}}\Lambda^\iota\big((R\cdot V_\omega)(\Lambda^{1/2}u)\big)||_2\, ds \lesssim \varepsilon_0\varepsilon_1 \int_0^t (1+s)(1+s)^{2\epsilon-1}(1+s)^{2\epsilon\delta}\, ds$$

$$\lesssim \varepsilon_0\varepsilon_1(1+t)^{2\epsilon(1+\delta)+1} \tag{3.6.33}$$

Controlling the second term in (3.6.30) is similar but we need to additionally use (3.C.1).

We start by using (3.C.2) and the bound $|\partial_\rho^k m_0(\rho)||\rho|^{k+1} \lesssim 1$ for any k, so we have:

$$\int_0^t ||e^{is\Lambda^{1/2}}\Lambda^{1/2+\iota}\big((m_0(\nabla)V_\omega)(\Lambda^{1/2}u)\big)||_2\,ds$$

$$\lesssim \int_0^t ||\Lambda^{-1}V_\omega(s)||_{L^{2/(1-2\epsilon)}}||\Lambda^{\iota+1}u(s)||_{L^{1/\epsilon}} + ||\Lambda^{-1/2+\iota}V_\omega(s)||_{L^{2/(1-2\epsilon)}}||\Lambda^{1/2}u(s)||_{L^{1/\epsilon}}\,ds, \tag{3.6.34}$$

Using fractional integration (3.C.1), we have $||\Lambda^{-1}V_\omega||_{L^{2/(1-2\epsilon)}} \lesssim ||V_\omega||_{L^{1/(1-\epsilon)}} \lesssim \varepsilon_1$ by (3.4.3). We also have $||\Lambda^{\iota+1}u(s)||_{L^{1/\epsilon}} \lesssim ||u(s)||_{W^{2,\infty}}^{1-2\epsilon}||u(s)||_{H^2}^{2\epsilon} \lesssim \varepsilon_0(1+s)^{2\epsilon(1+\delta)-1}$. The second term in (3.6.34) can be bounded in a nearly identical way and the result is that:

$$\int_0^t ||e^{is\Lambda^{1/2}}\Lambda^{1/2+\iota}\big((m_0(\nabla)V_\omega)(\Lambda^{1/2}u)\big)||_2\,ds \lesssim \varepsilon_0\varepsilon_1(1+t)^{2\epsilon(1+\delta)} \tag{3.6.35}$$

The third term in (3.6.30) contributes:

$$\int_0^t ||e^{is\Lambda^{1/2}}\Lambda^{-1/2+\iota}\big((R\cdot V_\omega(s))(\Lambda^{1/2}u(s))\big)||_2\,ds. \tag{3.6.36}$$

With $p_1 = 2(2-\iota)/3$, by (3.C.1), we have:

$$||\Lambda^{-1/2+\iota}\Big((R\cdot V_\omega)(\Lambda^{1/2}u)\Big)||_{L^2} \lesssim ||(R\cdot V_\omega)(\Lambda^{1/2}u)||_{L^{p_1}} \lesssim ||V_\omega||_{L^{p_1}}||u||_{W^{4,\infty}}, \tag{3.6.37}$$

so we have:

$$\int_0^t ||\Lambda^{-1/2+\iota}\big((R\cdot V_\omega(s))(\Lambda^{1/2}u(s))\big)||_2\,ds \lesssim \varepsilon_0\varepsilon_1\log(1+t) \tag{3.6.38}$$

The estimate for (3.6.31) follows after applying the fractional product rule (3.C.2) and (3.4.3), since $2/(1-2\epsilon) > 2$:

$$\int_0^t ||\Lambda^{1/2+\iota}\big((R\cdot xV_\omega)(\Lambda^{1/2}u)\big)||_{L^2}\,ds \lesssim \int_0^t ||xV_\omega(s)||_{W^{1,2/(1-2\epsilon)}}||u(s)||_{W^{1,1/\epsilon}}\,ds$$

$$\lesssim \varepsilon_0\varepsilon_1(1+t)^{2\epsilon(1+\delta)} \tag{3.6.39}$$

Summing up, we have shown that:

$$||\Lambda^\iota x g_3^I(t)||_2 \lesssim \varepsilon_0 \varepsilon_1 (1+t)^{2\epsilon(1+\delta)+1}, \tag{3.6.40}$$

and taking $\epsilon \leq \frac{\delta}{4(1+\delta)}$ gives the result. □

3.6.3 Estimates for g_4

We now bound the term which is quadratic in the vorticity:

$$g_4(t) = \int_0^t e^{is\Lambda^{1/2}} N(w,w)\, ds = \int_0^t e^{is\Lambda^{1/2}} \Lambda^{1/2+\iota} (R \cdot V_\omega(s))^2\, ds \tag{3.6.41}$$

We prove:

Lemma 3.6.3. *If v satisfies (3.2.9)-(3.2.11) with $\varepsilon_0 \ll 1$, then for $k \leq N_1 - 4$:*

$$||\nabla^k e^{-it\Lambda^{1/2}} g_4(t)||_{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon_1^2 (1+t) \tag{3.6.42}$$

$$||\Lambda^\iota x g_4(t)||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_1^2 (1+t)^2 \tag{3.6.43}$$

Proof. These estimates proceed in nearly exactly the same way as the estimates in the previous two lemmas. By Sobolev embedding and the fact that the Riesz transform is bounded on L^2:

$$||\nabla^k e^{-it\Lambda^{1/2}} g_4||_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t ||\nabla^k \Lambda^{1/2} (R \cdot V_\omega(s))^2||_{H^2(\mathbb{R}^2)}\, ds \lesssim \int_0^t ||V_\omega(s)||^2_{H^{k+3}(\mathbb{R}^2)}\, ds, \tag{3.6.44}$$

and by (3.4.3) and the bootstrap assumption (3.2.11), this implies (3.6.42).

Next, we compute:

$$i\mathcal{F}(xg_4)(\xi)$$

$$= \int_0^t e^{is|\xi|^{1/2}} \left(is\frac{\xi}{2|\xi|}\frac{\xi-\eta}{|\xi-\eta|} + m_1(\xi-\eta)|\xi|^{1/2} + \frac{\xi}{2|\xi|^{3/2}}\frac{\xi-\eta}{|\xi-\eta|} \right) \hat{V}_\omega(\xi-\eta)\frac{\eta}{|\eta|}\hat{V}_\omega(\eta)\, d\eta ds$$

$$+ \int_0^t e^{is|\xi|^{1/2}}|\xi|^{1/2}\frac{\xi-\eta}{|\xi-\eta|}\frac{\eta}{|\eta|}\partial_\xi \hat{V}_\omega(\xi-\eta)\hat{V}_\omega(\eta)\, d\eta ds, \quad (3.6.45)$$

again with $m_1(\xi-\eta) = \partial_\xi(|\xi-\eta|^{-1}(\xi-\eta))$. We have:

$$\int_0^t s||\Lambda^\iota R(RV_\omega)^2||_{L^2(\mathbb{R}^2)}\, ds \lesssim \int_0^t s||V_\omega(s)||^2_{H^3(\mathbb{R}^2)}\, ds \lesssim \varepsilon_1^2(1+t)^2. \quad (3.6.46)$$

Using fractional integration (3.C.1), the fact that the Riesz transform is bounded on L^4, and the fractional product rule (3.C.2):

$$\int_0^t ||\Lambda^\iota(m_1(\nabla)V_\omega)(R\cdot V_\omega)||_{L^2}\, ds \lesssim \int_0^t ||\Lambda^{-1}V_\omega(s)||_{W^{1,4}}||V_\omega(s)||_{W^{1,4}}\, ds$$

$$\lesssim \int_0^t ||V_\omega(s)||_{W^{1,4/3}}||V_\omega||_{W^{1,4}}\, ds \lesssim \varepsilon_1^2(1+t), \quad (3.6.47)$$

after using Proposition 3.4.3. Similarly:

$$\int_0^t ||\Lambda^{-1/2+\iota}(R\cdot xV_\omega)(R\cdot V_\omega)||_{L^2}\, ds \lesssim \int_0^t ||(R\cdot xV_\omega)(R\cdot V_\omega)||_{L^{p_1}}\, ds$$

$$\lesssim \int_0^t ||xV_\omega||_{8/3}||V_\omega||_{p_2}\, ds \lesssim \varepsilon_1^2(1+t), \quad (3.6.48)$$

where $p_1 = 2(2-\iota)/4$ and p_2 satisfies $1/p_2 + 3/8 = 1/p_1$.

□

3.6.4 Estimates for g_5

Recall that g_5 contains all terms of order three or higher which involve V_ω. There are two such types of terms: the terms coming from the first line of (3.3.59), and the terms of degree 2 and higher from expanding the rescaled Dirichlet-to-Neumann map $G(h)$ in powers of h and inserting this into (3.3.59). In either case, the vorticity enters at most quadratically.

We prove:

Proposition 3.6.2. *If (v, h) satisfy (3.2.9)-(3.2.11) for $\varepsilon_1 \ll \varepsilon_0 \ll 1$, then for $I = 1, 2$:*

$$(1+t)||\nabla^k e^{-it\Lambda^{1/2}} g_5||_{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon_0^2 + \varepsilon_1(1+t)^2, \tag{3.6.49}$$

$$(1+t)^{-\delta}||xg_5||_{L^2(\mathbb{R}^2)} \lesssim \varepsilon_0^2 + \varepsilon_1(1+t)^{2-\delta} \tag{3.6.50}$$

Proof. We first illustrate one of the terms coming from the first line of (3.3.59), the other terms coming from this line being similar. We will just bound the term corresponding to the first term on the right-hand side of (3.3.59), which is:

$$R_1 = -\int_0^t e^{i(s-t)\Lambda^{1/2}} |V_\omega \cdot \Lambda^{-1/2}\nabla u|^2\, ds, \tag{3.6.51}$$

By Sobolev embedding:

$$||\nabla^k R_1||_{L^\infty(\mathbb{R}^2)} \lesssim ||R_1||_{H^{k+2}(\mathbb{R}^2)} \lesssim \int_0^t ||V_\omega(s)||^2_{H^{k+2}(\mathbb{R}^2)} ||u(s)||^2_{W^{k+3,\infty}(\mathbb{R}^2)}, \tag{3.6.52}$$

Using the interpolation inequality (3.A.8) as in the proof of Lemma 3.6.2, we $||u(s)||_{W^{1/\epsilon,k+3}(\mathbb{R}^2)} \lesssim \varepsilon_0(1+t)^{-1+\sigma}$, where $\sigma = \epsilon + \frac{k+3}{N-1+\epsilon}(1-\epsilon+\delta)$. Therefore:

$$||\nabla^k R_1||_{L^\infty(\mathbb{R}^2)} \lesssim \int_0^t \varepsilon_0^2\varepsilon_1^2(1+s)^{-2+2\sigma}(1+s)^{2\delta}\, ds \lesssim \varepsilon_0^2\varepsilon_1^2(1+t)^{-1+2\sigma}(1+t)^{2\delta}, \tag{3.6.53}$$

and this is clearly bounded by the right-hand side of (3.6.49) for δ sufficiently small and N_0 sufficiently large. The estimates for $\Lambda^\iota x R_1$ are similar to this and the estimates in the previous three sections.

The terms coming from the higher-order terms expansion of the Dirichlet-to- Neumann map in powers of h can be bounded by using the estimates from Appendix F of [4]. See Section 3.8. $\square$

3.7 Estimates for the dispersive terms

In this section we bound the term g_1 defined in (3.3.76). We proceed nearly exactly as in [4] to handle these terms. The only differences here are that: (1) after performing the normal forms transformation (integration by parts in time), there are additional terms involving the vorticity that need to be bounded and, (2) we want to control $||\Lambda^{\iota} x g_1||_{L^2(\mathbb{R}^2)}$ instead of $||x g_1||_{L^2(\mathbb{R}^2)}$. Both of these points are very simple but tedious to deal with; point (1) will involve estimates very similar to those in the previous section and point (2) will involve superficial changes to the proofs in [4] and we begin by recalling the setup used in that paper.

Before proceeding it is also helpful to recall some of the terminology from [4]. The following definitions and theorems are nearly verbatim from that paper and we include them here for the convenience of the reader. From now on, we will also write $||u||_p = ||u||_{L^p(\mathbb{R}^2)}$. Given a function $m : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, define:

$$\mathcal{F}(B_m(u,v)(\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m(\xi,\eta)\hat{u}(\eta)\hat{v}(\xi-\eta)\, d\eta, \tag{3.7.1}$$

and given a function $m : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, define:

$$\mathcal{F}(T_m(u,v,w))(\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m(\xi,\eta,\sigma)\hat{u}(\sigma)\hat{v}(\eta)\hat{w}(\xi-\eta-\sigma)\, d\sigma d\eta. \tag{3.7.2}$$

We note that we can think of any of the bilinear multipliers $m(\xi,\eta)$ as functions of any of two of the three variables $\xi, \eta, \xi-\eta$. Consequently it is helpful to adopt the notation

$\xi_1 = \xi, \xi_2 = \eta, \xi_3 = \xi - \eta$. We will encounter bilinear operators B_m with symbols m in the following class:

Definition 1. If $m : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, we say $m \in \mathcal{B}_s$ if:

- it is homogeneous of order s,
- it is smooth outside of $\{\xi_i = 0\}, i = 1, 2, 3$,
- for $i = 1, 2, 3$, if $|\xi_i| \ll |\xi_{i+1}|$ and $|\xi_{i+2}| \sim 1$, then there is a smooth function $\mathcal{A}$ so that $m = \mathcal{A}(|\xi_i|^{1/2}, \xi_i/|\xi|, \xi_{i+1})$, with $\xi_4 = \xi, \xi_5 = \xi - \eta$.

This class is larger than the "standard" class of bilinear multipliers which satisfy the hypotheses of the Coifman-Meyer theorem, since they are allowed to have singularities along the "axes" $\xi = 0, \eta = 0$ and $\xi - \eta = 0$.

We will further say that $m \in \tilde{\mathcal{B}}_s$ if $m \in \mathcal{B}_s$ and $\operatorname{supp} m \subset \{|\eta| \gtrsim |\xi|\}$. Then we have:

Theorem 3.7.1 (Theorem C.1 from [4]). *If $m \in \mathcal{B}_0$ then:*

$$||B_m(u, v)||_r \lesssim ||u||_p ||v||_q, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, 1 < p, q < \infty \tag{3.7.3}$$

If $m \in \tilde{\mathcal{B}}_s$ and $k \in \mathbb{Z}$, then:

$$||\nabla^k B_m(u, v)||_r \lesssim ||\Lambda^{k+s} u||_p ||v||_q, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{3.7.4}$$

We now consider trilinear operators. The class of trilinear operators that need to be considered in this problem has a fairly complicated definition which we will omit here. This is because, in addition to having Coiffman-Meyer type singularities, they can have singularities along higher-dimensional subsets of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. See Definition D.1 of [4] for the definition of the classes $\mathcal{T}_s, \widetilde{\mathcal{T}}_s$, used in the next theorem, which is the analog of Theorem3.7.1:

Theorem 3.7.2 (Theorem D.1 in [4]). *If $m \in \mathcal{T}_0$, then if $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$ and $1 < p_1, p_2, p_3 < \infty$:*

$$||B_m(u,v,w)||_r \lesssim ||u||_{p_1}||v||_{p_2}||w||_{p_3}, \tag{3.7.5}$$

and if $m \in \widetilde{\mathcal{T}}_s$,

$$||\nabla^k T_m(u,v,w)||_r \lesssim ||\Lambda^{k+s}u||_{p_1}||v||_{p_2}||w||_{p_3}, \tag{3.7.6}$$

Finally, if $0 < \alpha < 2$ and $m \in \mathcal{T}_0$ then for $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{\alpha}{2} = \frac{1}{r}$:

$$||T_{m|\xi-\eta|^{-\alpha}}(u,v,w)||_r \lesssim ||u||_{p_1}||v||_{p_2}||w||_{p_3}, \tag{3.7.7}$$

As in [4] the point of (3.7.7) is that if m were 1, then we would have $T_{m|\xi-\eta|^{-\alpha}}(u,v,w) = v\Lambda^{-\alpha}(uw)$, which explains the relation between the exponents.

We start by re-writing and higher-order terms $B(u), T(u), R(u)$ from Proposition 3.3.1. We write:

$$B = \sum_{j=1}^{2} \sum_{\alpha,\beta\in\{+,-\}} c_{j\alpha\beta} g_{j\alpha\beta}, \qquad T = \sum_{j=1}^{2} \sum_{\alpha\beta\gamma\in\{+,-\}} c_{j\alpha\beta,\gamma} g_{j\alpha\beta\gamma}, \tag{3.7.8}$$

for constants $c_{j\alpha\beta}$ and $c_{j\alpha\beta\gamma}$ and where, taking the Fourier transform $\mathcal{F}$:

$$\mathcal{F}g_{j\alpha\beta}(\xi) = \int_0^t \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta}(\xi,\eta)} m_j(\xi,\eta) \hat{f}_{-\alpha}(s,\xi-\eta)\hat{f}_{-\beta}(s,\eta)\, d\eta ds \tag{3.7.9}$$

for complex numbers $c_{j\alpha\beta}$, with $\hat{f}_+ = \hat{f}, \hat{f}_- = \overline{\hat{f}}$, where $\varphi_{\pm\pm} = |\xi|^{1/2} \pm |\eta|^{1/2} \pm |\xi-\eta|^{1/2}$ and:

$$m_1(\xi,\eta) = |\eta|^{-1/2}(\xi\cdot\eta - |\xi||\eta|), \tag{3.7.10}$$

$$m_2(\xi,\eta) = \frac{1}{2}\frac{|\xi|^{1/2}}{|\eta|^{1/2}|\xi-\eta|^{1/2}}(\eta\cdot(\xi-\eta) + |\eta||\xi-\eta|). \tag{3.7.11}$$

Similarly, we are writing:

$$\mathcal{F}g_{j\alpha\beta\gamma}(\xi) = \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta\gamma}(\xi,\eta,\sigma)} m_j(\xi,\eta,\sigma) \hat{f}_{-\alpha}(s,\sigma)\hat{f}_{-\beta}(s,\eta)\hat{f}_{-\gamma}(s,\xi-\eta-\sigma)\, d\sigma d\eta ds, \tag{3.7.12}$$

for constants $c_{j\alpha\beta\gamma}$, with $\varphi_{\pm\pm\pm} = |\xi|^{1/2} \pm |\eta|^{1/2} \pm |\sigma|^{1/2} \pm |\xi-\eta-\sigma|^{1/2}$ and:

$$m_1(\xi,\eta,\sigma) = -\frac{1}{2}|\xi|\big(|\xi-\eta-\sigma|^{3/2} + |\xi||\xi-\eta-\sigma|^{1/2} - 2|\xi-\eta||\xi-\eta-\sigma|^{1/2}\big) \tag{3.7.13}$$

$$m_2(\xi,\eta,\sigma) = |\xi|^{1/2}|\eta|^{1/2}\big(|\xi-\eta-\sigma|^{3/2} - |\xi-\eta||\xi-\eta-\sigma|^{1/2}\big), \tag{3.7.14}$$

To handle the bilinear terms, we integrate by parts in time; for each $j = 1,2$, and $\alpha,\beta \in \{+,-\}$, with $\mu_{j\alpha\beta} = \frac{1}{i\varphi_{\alpha\beta}} m_j$ we have:

$$\begin{aligned}\mathcal{F}g_{j\alpha\beta}(\xi) = \mathcal{F}g^1_{j\alpha\beta}(\xi) - \mathcal{F}g^2_{j\alpha\beta}(\xi) &= \int_{\mathbb{R}^2} e^{it\varphi_{\alpha\beta}(\xi,\eta)} \mu_{j\alpha\beta}(\xi,\eta)\hat{f}_{-\alpha}(t,\xi-\eta)\hat{f}_{-\beta}(t,\eta)\, d\eta \\ &\quad - \int_0^t \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta}(\xi,\eta)} \mu_{j\alpha\beta}(\xi,\eta)\partial_s\big(\hat{f}_{-\alpha}(s,\xi-\eta)\hat{f}_{-\beta}(s,\eta)\big)\, d\eta ds \end{aligned} \tag{3.7.15}$$

Recalling the Duhamel form of the equations (3.1.38), we further write:

$$\begin{aligned}\mathcal{F}g^2_{j\alpha\beta}(\xi) &= \int_0^t \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta}} \mu_{j\alpha\beta}(\xi,\eta)\big(\hat{f}^1_{-\alpha}(s,\xi-\eta)\hat{f}_{-\beta}(s,\eta) + \hat{f}_{-\alpha}(s,\xi-\eta)\hat{f}^1_{-\beta}(s,\eta)\big)\, d\eta \\ &\quad + \int_0^t \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta}} \mu_{j\alpha\beta}(\xi,\eta)\big(\hat{f}^2_{-\alpha}(s,\xi-\eta)\hat{f}_{-\beta}(s,\eta) + \hat{f}_{-\alpha}(s,\xi-\eta)\hat{f}^2_{-\beta}(s,\eta)\big)\, d\eta \\ &\equiv \mathcal{F}g_{j\alpha\beta}(\xi) + \mathcal{F}A_\omega(\xi), \end{aligned} \tag{3.7.16}$$

the point being that $g^1_{j\alpha\beta}(\xi)$ is trilinear and only involves the purely dispersive variable u while A_ω consists of all the terms contributed by the vorticity. Expanding out $g_{j\alpha\beta}(\xi)$, we

have:

$$g_{j\alpha\beta} = \sum_{\gamma\in\{\pm\}} \sum_{k=1,2} g^1_{jk\alpha\beta\gamma} + g^2_{jk\alpha\beta\gamma}, \tag{3.7.17}$$

where:

$$\mathcal{F}g^1_{jk\alpha\beta\gamma} = \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta\gamma}} \mu_{j\alpha\beta}(\xi,\eta) m_k(\xi-\eta,\sigma)$$
$$\times \hat{f}_{-\alpha}(s,\xi-\eta-\sigma)\hat{f}_{-\beta}(s,\sigma)\hat{f}_{-\gamma}(s,\eta) d\eta d\sigma ds, \tag{3.7.18}$$

$$\mathcal{F}g^2_{jk\alpha\beta\gamma} = \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta\gamma}} \mu_{j\alpha\beta}(\xi,\xi-\eta) m_k(\xi-\eta,\sigma)$$
$$\times \hat{f}_{-\alpha}(s,\xi-\eta-\sigma)\hat{f}_{-\beta}(s,\sigma)\hat{f}_{-\gamma}(s,\eta) d\eta d\sigma ds, \tag{3.7.19}$$

where the bilinear multipliers m_j, m_k are defined in (3.7.11).

Finally, when $(\alpha,\beta,\gamma) \in \{(+,+,+),(-,+,+),(+,-,+),(+,+,-),(-,-,-)\}$, we will integrate by parts again in the trilinear terms $T_{j\alpha\beta\gamma}$ and $T_{jk\alpha\beta\gamma}$ (for the other values of (α,β,γ), the phase $\varphi_{\alpha\beta\gamma}$ vanishes on too large a set for this strategy to work). We therefore write $\mu_{j\alpha\beta\gamma} = \frac{1}{i\varphi_{\alpha\beta\gamma}} m_j$ as well as:

$$\mathcal{F}g_{j\alpha\beta\gamma} = \mathcal{F}g^1_{j\alpha\beta\gamma} + \mathcal{F}g^2_{j\alpha\beta\gamma} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\varphi_{\alpha\beta\gamma}} \mu_{j\alpha\beta\gamma} \hat{f}(t,\xi-\eta-\sigma)\hat{f}(t,\sigma)\hat{f}(t,\eta)\, d\sigma d\eta$$
$$- \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{is\varphi_{\alpha\beta\gamma}} \mu_{j\alpha\beta\gamma} \partial_s \big(\hat{f}(s,\xi-\eta-\sigma)\hat{f}(s,\sigma)\hat{f}(s,\eta)\big)\, d\sigma d\eta ds, \tag{3.7.20}$$

with a similar definition for $g^1_{jk\alpha\beta\gamma}, g^2_{jk\alpha\beta\gamma}$.

With the above terminology, we can now prove estimates for these terms. The next lemma follows directly from the estimates in [4]:

Lemma 3.7.1. *For each $j = 1, 2$ and $\alpha, \beta \in \{+, -\}$, if $||u||_X \leq \varepsilon_0$ then:*

$$||e^{-it\Lambda^{1/2}} g^1_{j\alpha\beta}||_{W^{4,\infty}} \lesssim \frac{\varepsilon_0^2}{1+t}. \tag{3.7.21}$$

For $(\alpha, \beta, \gamma) \notin \{(+,-,-), (-,+,-), (-,-,+)\}$ and $1 \leq j, k \leq 2$:

$$||e^{-it\Lambda^{1/2}} g^1_{j\alpha\beta\gamma}||_{W^{4,\infty}} + ||e^{-it\Lambda^{1/2}} g^1_{jk\alpha\beta\gamma}||_{W^{4,\infty}} \lesssim \frac{\varepsilon_0^3}{1+t} \tag{3.7.22}$$

Estimates for the weighted part of the norm can be proven in a nearly identical way to the corresponding estimates in [4]. Since we control $\Lambda^\iota(xf)$ in L^2 for small ι instead of xf in L^2, the argument becomes somewhat more tedious because we must repeatedly use the fractional integration estimate 3.C.1. We include the details here for the convenience of the reader and for the sake of completeness, but we emphasize that the estimates in the remainder of this section do not involve any ideas which are not already present in [4].

Lemma 3.7.2. *For each $j = 1, 2$ and $\alpha, \beta \in \{+, -\}$, if $||u||_X \leq \varepsilon$ then:*

$$||\Lambda^\iota x g_{j\alpha\beta}||_2 \lesssim \varepsilon_0^2. \tag{3.7.23}$$

For $1 \leq j, k \leq 2$ and $(\alpha, \beta, \gamma) \notin \{(+,-,-), (-,+,-), (-,-,+)\}$ we also have:

$$||\Lambda^\iota x g_{j\alpha\beta\gamma}||_2 + ||\Lambda^\iota x g_{jk\alpha\beta\gamma}||_2 \lesssim \varepsilon_0^2. \tag{3.7.24}$$

Proof. We start with (3.7.23). Fix j, α, β and write $\mu = \mu_{j\alpha\beta}$. In what follows we will only use the fact that $\mu \in \mathcal{B}_1$. Let χ_1 be a cutoff function which is homogeneous of degree zero and so that $\chi_1(\xi, \eta) = 0$ when $|\eta| \leq 11/10|\xi|$ and $\chi_1(\xi, \eta) = 1$ when $|\eta| \geq 9/10|\xi|$, say. Set $\chi_2 = 1 - \chi_1$. Then it is enough to prove the estimate with μ replaced by $\mu_1 = \chi_1\mu$ and $\mu_2 = \chi_2\mu$ and by symmetry it is therefore enough to prove the estimate for μ_1. Taking the

Fourier transform, we write:

$$\mathcal{F}(xB_{\mu_1}) = i\partial_\xi B_{\mu_1}$$

$$= \int_{\mathbb{R}^2} e^{it\varphi(\xi,\eta)}\mu_1(\xi,\eta)\hat{f}(t,\eta)\partial_\xi\hat{f}(t,\xi-\eta)\,d\eta + \int_{\mathbb{R}^2} e^{it\varphi(\xi,\eta)}\partial_\xi\mu_1(\xi,\eta)\hat{f}(t,\eta)\hat{f}(t,\xi-\eta)\,d\eta$$

$$+ \int_{\mathbb{R}^2} it\partial_\xi\varphi(\xi,\eta)e^{it\varphi(\xi,\eta)}\mu_1(\xi,\eta)\hat{f}(t,\eta)\hat{f}(t,\xi-\eta)\,d\eta \equiv I + II + III. \quad (3.7.25)$$

To control $||\Lambda^\iota I||_2$, we take $p = p(\iota)$ so that $-\iota = 2/p - 1$ (thus $p = 2 + O(\iota)$) and then q so that $1/p + 1/q = 3/4$ (thus $q = 4 - O(\iota)$). We also note that because μ^1 is supported on the set $|\xi| \leq 9/10|\eta|$, we have $|\xi|^\iota|m(\xi,\eta)| \lesssim |\eta|^\iota|m(\xi,\eta)|$, so, using Sobolev embedding and the multiplier estimate (3.7.3):

$$||\Lambda^\iota I||_2 = ||\Lambda^\iota B_{\mu^1}(f, xf)||_2 \lesssim ||B_{\mu^1}(\Lambda^\iota u, xf)||_2$$

$$\lesssim ||B_{\mu^1}(\Lambda^\iota u, \Lambda^{-\iota}\Lambda^\iota xf)||_{W^{1,4/3}} \lesssim ||u||_{W^{1+\iota,q}}||\Lambda^\iota xf||_2 \lesssim (1+t)^\delta||u||_X^2. \quad (3.7.26)$$

To control II, we use the fact that μ^2 is a symbol in $\mathcal{B}_1$ and also that μ^2 vanishes to order $1/2$ at $\xi = 0$, from which it follows that:

$$\partial_\xi\mu_1 = \nu_0 + |\xi|^{-1/2}\nu_1 + |\xi-\eta|^{-1}\nu_2, \qquad \nu_0 \in \widetilde{\mathcal{B}}_0, \nu_1 \in \widetilde{\mathcal{B}}_{1/2}, \nu_2 \in \widetilde{\mathcal{B}}_1. \quad (3.7.27)$$

Plugging this decomposition into the definition of II, the contribution of ν_0 is easy to handle so we skip it. With $p_0 = p_0(\iota)$ satisfying $1/2 - \iota = 2/p_0 - 1$ (thus $p_0 = 4/3 + O(\iota)$), we have:

$$||\Lambda^\iota\Lambda^{-1/2}B_{\nu_1}(u,u)||_2 \lesssim ||B_{\nu_1}(\Lambda^\iota u, u)||_{p_0} \lesssim ||\Lambda^{1/2+\iota}u||_{p_1}||u||_2 \lesssim (1+t)^\delta||u||_X^2, \quad (3.7.28)$$

where p_1 satisfies $1/p_1 + 1/2 = 1/p_0$ (thus $p_1 = 4 - O(\iota)$). To hande the third term from

(3.7.27):

$$||\Lambda^\iota B_{\nu_2}(u, e^{it\Lambda^{1/2}}\Lambda^{-1}xf)||_2 \lesssim ||\Lambda^{1+\iota}u||_{p_2}||\Lambda^{-1-\iota}\Lambda^\iota e^{it\Lambda^{1/2}}f||_{q_2} \lesssim ||\Lambda^{1+\iota}u||_{p_2}||f||_{4/3}, \quad (3.7.29)$$

where here q_2 is taken so that $1-\iota = 2/q_2 - 2/(4/3)$ (thus $q_2 = 4+O(\iota)$), $1/p_2 + 1/q_2 = 1/2$ (thus $p_2 = 4 - O(\iota)$). Since $||f||_{4/3} \lesssim ||f||_2 + ||xf||_2$, this is also bounded by $(1+t)^\delta ||u||_X^2$.

Finally, to control $||\Lambda^\iota III||_2$, we start by noting that since $\partial_\xi \varphi = \frac{1}{2}\frac{\xi}{|\xi|^{3/2}} \pm \frac{1}{2}\frac{\xi-\eta}{|\xi-\eta|^{3/2}}$, we have:

$$\partial_\xi \phi(\xi,\eta)\mu^2(\xi,\eta) = \nu_0'(\xi,\eta) + \frac{1}{|\xi-\eta|^{-1/2}}\nu_1'(\xi,\eta), \qquad \nu_0' \in \widetilde{\mathcal{B}}_{1/2}, \nu_1' \in \widetilde{\mathcal{B}}_1. \quad (3.7.30)$$

The contribution from the first term can be handled using similar arguments to the above, and the contribution from the second term can be handled using fractional integration:

$$||B_{|\xi-\eta|^{-1/2}|\xi|^\iota \nu_1'}(u, e^{\pm it\Lambda^{1/2}}f)||_2 = ||\Lambda^{1+\iota}u||_{1/\delta_0}||e^{\pm it\Lambda^{1/2}}\Lambda^{-1/2}f||_{2/(1-2\delta_0)}$$

$$\lesssim ||\Lambda^{1+\iota}u||_{1/\delta_0}||f||_{4/(3-4\delta_0)}. \quad (3.7.31)$$

By (3.A.8), the first factor is bounded by $\varepsilon_0(1+t)^{-1+a}$ with $a = O(\delta_0)$, and to bound the second factor we use Hölder's inequality to control it by:

$$||f||_{4/(3-4\delta_0)} \lesssim ||xf||_{L^{p_1}} \lesssim ||\Lambda^\iota xf||_{L^2}, \quad (3.7.32)$$

with $p_1 = \frac{2}{1-\iota}$, where we have used fractional integration (3.C.1) again.

We now handle the weakly resonant cubic terms. The arguments are very similar to the above but rely on the trilinear estimates (3.7.5)-(3.7.6) in place of the bilinear estimates (3.7.3)-(3.7.4). We fix j, α, β, γ as in the statement of the theorem and let χ_1, χ_2, χ_3 a partition of unity on $(\mathbb{R}^2)^3$ so that $\chi_1(\xi,\eta,\sigma)$ is supported away from $\{\sigma = 0\}$, $\chi_2(\xi,\eta,\sigma)$ is supported away from $\{\eta = 0\}$ and $\chi_3(\xi,\eta,\sigma)$ is supported away from $\{\xi-\eta-\sigma = 0\}$. It then suffices to prove the estimates with μ replaced by $\mu_\ell = \mu\chi_\ell$, $\ell = 1,2,3$. We will just prove the estimates for μ_3, the others being similar.

We start by writing:

$$i\mathcal{F}(xT_{\mu_3}) = \partial_\xi\left(\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\varphi(\xi,\eta,\sigma)}\mu_3(\xi,\eta,\sigma)\hat{f}(s,\sigma)\hat{f}(s,\eta)\hat{f}(s,\xi-\eta-\sigma)\,d\sigma d\eta\right) \tag{3.7.33}$$

$$= \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\varphi(\xi,\eta,\sigma)}\mu_3(\xi,\eta,\sigma)\hat{f}(s,\sigma)\hat{f}(s,\eta)\partial_\xi\hat{f}(s,\xi-\eta-\sigma)\,d\sigma d\eta \tag{3.7.34}$$

$$+ \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{it\varphi(\xi,\eta,\sigma)}\partial_\xi\mu_3(\xi,\eta,\sigma)\hat{f}(s,\sigma)\hat{f}(s,\eta)\hat{f}(s,\xi-\eta-\sigma)\,d\sigma d\eta \tag{3.7.35}$$

$$+ \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} ite^{it\varphi(\xi,\eta,\sigma)}\partial_\xi\varphi(\xi,\eta,\sigma)\mu_3(\xi,\eta,\sigma)\hat{f}(s,\sigma)\hat{f}(s,\eta)\hat{f}(s,\xi-\eta-\sigma)\,d\sigma d\eta \tag{3.7.36}$$

$$\equiv I + II + III. \tag{3.7.37}$$

By Sobolev embedding, the trilinear multiplier estimate (3.7.6) and the interpolation inequality (3.A.8), we have:

$$||\Lambda^\iota I||_2 \lesssim \int_0^t ||T_{\mu_3}(\Lambda^\iota u, u, e^{\pm it\Lambda^{1/2}}xf)||_2\,ds \lesssim \int_0^t ||T_{\mu_3}(\Lambda^\iota u, u, e^{\pm it\Lambda^{1/2}}xf)||_{4/3}\,ds$$
$$\lesssim \int_0^t ||\Lambda^{5/2+\iota}u||_{W^{1,8}}||u||_{p_1}||\Lambda^{-\iota}\Lambda^\iota xf||_{p_2}\,ds \lesssim (1+t)^\delta||u||_X^3, \tag{3.7.38}$$

where $1/p_1 + 1/p_2 = 3/4$ with p_1 chosen so that $-\iota = 2/p_1 - 1$ (thus $p_1 = 2 + O(\iota), p_2 = 8 - O(\iota)$).

To handle II, we note that:

$$\partial_\xi\mu^2 = \nu_0 + \frac{1}{|\xi|^{1/2}}\nu_1 + \frac{1}{|\xi-\eta|^{1/2}}\nu_2 + \frac{1}{|\xi-\eta-\sigma|}\nu_3, \quad \nu_0 \in \widetilde{T}_{3/2}, \nu_1, \nu_2 \in \widetilde{T}_2, \nu_3 \in \widetilde{T}_{5/2}. \tag{3.7.39}$$

The contribution from ν_0 is straightforward to bound so we ignore it. To handle the contribution from the third term, we argue as in the corresponding estimate for the bilinear terms; with p_0 satisfying $1/2 - \iota = 2/p_0 - 1$ (thus $p_0 = 4/3 + O(\iota)$):

$$\int_0^t ||\Lambda^{-1/2+\iota} T_{\nu_1}(u,u,u)||_2 \, ds \lesssim \int_0^t ||T_{\nu_1}(u,u,u)||_{p_0} \, ds \lesssim \int_0^t ||\Lambda^2 u||_4 ||u||_4^2 \, ds \lesssim (1+t)^\delta ||u||_X^3. \tag{3.7.40}$$

To bound the contribution from ν_2, we use (3.7.7):

$$\int_0^t ||\Lambda^\iota T_{|\xi-\eta|^{-1/2}\nu_2}(u,u,u)||_2 \, ds \lesssim \int_0^t ||\Lambda^{2+\iota} u||_8 ||u||_8^2 \, ds \lesssim (1+t)^\delta ||u||_X^3. \tag{3.7.41}$$

The contribution from ν_3 can be bounded in a similar manner to (3.7.29).

Finally we bound the contribution from III from (3.7.37). We start with the fact that $\partial_\xi \phi = \frac{1}{2} \frac{\xi}{|\xi|^{3/2}} \pm \frac{\xi-\eta-\sigma}{|\xi-\eta-\sigma|^{3/2}}$ and so III becomes:

$$\begin{aligned} III &= \frac{1}{2} \int_0^t e^{is\varphi} s \frac{\xi}{|\xi|^{3/2}} \mu_3(\xi,\eta,\sigma) \hat{f}(s,\sigma) \hat{f}(s,\eta) \hat{f}(s,\xi-\eta-\sigma) \, ds d\sigma d\eta \\ &+ \frac{1}{2} \int_0^t e^{is\varphi} s \frac{\xi-\eta-\sigma}{|\xi-\eta-\sigma|^{3/2}} \mu_3(\xi,\eta,\sigma) \hat{f}(s,\sigma) \hat{f}(s,\eta) \hat{f}(s,\xi-\eta-\sigma) \, ds d\sigma d\eta = III_a + III_b. \end{aligned} \tag{3.7.42}$$

Note that μ_3 vanishes to at least order $1/2$ at $\xi = 0$ which implies that $\frac{\xi}{|\xi|^{3/2}} \mu_3 \in \widetilde{T}_2$. Therefore:

$$||\Lambda^\iota III_a||_2 \lesssim \int_0^t s ||T_{\frac{\xi}{|\xi|^{3/2}} \mu_3}(u,u,u)||_2 \, ds \lesssim \int_0^t s ||\Lambda^{2+\iota} u||_6 ||u||_6^2 \, ds \lesssim (1+t)^\delta ||u||_X^3. \tag{3.7.43}$$

To control III_b, we use fractional integration again; with δ_0 a small constant, we have:

$$\begin{aligned} ||\Lambda^\iota III_b||_2 &\lesssim \int_0^t s ||T_{|\xi|^\iota \frac{\xi-\eta-\sigma}{|\xi-\eta-\sigma|^{3/2}} \mu_3}(u,u,e^{\pm is\Lambda^{1/2}} \Lambda^{-1/2} f)||_2 \\ &\lesssim \int_0^t s ||\Lambda^{5/2+\iota} u||_{1/\delta_0} ||u||_{1/\delta_0} ||e^{\pm is\Lambda^{1/2}} \Lambda^{-1/2} f||_{4/(3-8\delta_0)} \, ds. \end{aligned} \tag{3.7.44}$$

Using point (3) from Lemma A.1 from [4], we bound the last factor by $||f||_p$ where p satisfies $1/2 = 2/(3-8\delta) - 2/p$, and then bound $||f||_p \lesssim ||xf||_r = ||\Lambda^{-\iota}\Lambda^{\iota}xf||_r \lesssim ||\Lambda^{\iota}xf||_2$, where $r > 2$ is picked so that $\iota = 1 - 2/r$. Using (3.2.9)-(3.2.11), this shows that (3.7.44) is bounded by $\varepsilon_0^3(1+t)^{\delta}$, as required.

$\square$

It now remains to prove estimates for the cubic terms not covered by (3.7.24) (the "strongly resonant cubic terms" in the terminology of [4]). Fixing $\mu = \mu_{j\alpha\beta\gamma}$ or $\mu_{jk\alpha\beta\gamma}$ with $1 \leq j,k \leq 2$, $(\alpha,\beta,\gamma) \in \{(+,-,-),(-,+,-),(-,-,+)\}$, write:

$$g = \int_0^t G(s)\,ds, \quad \mathcal{F}G(s,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{is\varphi}\mu(\xi,\eta,\sigma)\hat{f}(s,\xi-\eta-\sigma)\hat{f}(s,\eta)\hat{f}(s,\sigma)\,d\sigma d\eta. \tag{3.7.45}$$

As in the above lemma it will suffice to prove the estimates after replacing μ with $\mu_3 = \mu\chi_3$ where χ_3 is a cutoff function supported near $\{\xi = \sigma\}$. We let $\Theta(\sigma)$ be a cutoff function which is one when $|\sigma| \leq 1/2$ and which is zero for $|\sigma| \geq 1$, fix a small parameter δ_0, and further decompose $g = g_{low} + g_{high}$, where:

$$g_{low} = \int_0^t G_{low}(s)\,ds, \text{ where} \tag{3.7.46}$$

$$\mathcal{F}G_{low}(s,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{is\varphi}\mu_3(\xi,\eta,\sigma)\Theta(\sigma/s^{\delta_0})\hat{f}(s,\xi-\eta-\sigma)\hat{f}(s,\eta)\hat{f}(s,\sigma)\,d\sigma d\eta. \tag{3.7.47}$$

Also define $\widetilde{G}_{low}$ by:

$$xG_{low} = e^{is\Lambda^{1/2}}\widetilde{G}_{low}. \tag{3.7.48}$$

It turns out that the hardest task is to get estimates for $\widetilde{G}_{low}$, and the main technical result in Chapter 7 of [4] is:

Lemma 3.7.3. *There is a constant $\kappa_1 = \kappa_1(\delta_0, N) > 0$ so that if δ_0 is taken sufficiently small, then*

$\tilde{G}_{low}$ satisfies the following estimates:

$$||\Lambda^{\iota}\tilde{G}_{low}||_2 + ||\tilde{G}_{low}||_2 \lesssim s^{-1-\kappa_1}||u||_X^3, \tag{3.7.49}$$

and, provided δ_1 is taken sufficiently small:

$$||\Lambda^{\iota}\tilde{G}_{low}||_{2-\delta_1} + ||\tilde{G}_{low}||_{2-\delta_1} \lesssim s^{-1-\kappa_1/2}||u||_X^3. \tag{3.7.50}$$

When $\iota = 0$, this is the content of the estimate after (7.10c) in [4] and it is not hard to see that the same argument given there, combined with the arguments in the previous lemma gives the result for small $\iota > 0$ as well, after possibly taking κ_1 smaller.

With the estimates (3.7.49)-(3.7.50), the rest of the arguments from Chapter 7 of [4] go through without change which proves:

Lemma 3.7.4. *If $1 \leq j, k \leq 2$ and $(\alpha, \beta, \gamma) \in \{(-,-,+), (-,+,-), (+,-,-)\}$, then:*

$$||g_{j\alpha\beta\gamma}||_{W^{4,\infty}} + ||g_{jk\alpha\beta\gamma}||_{W^{5,\infty}} \lesssim \frac{\varepsilon_0^3}{1+t}, \tag{3.7.51}$$

and

$$||\Lambda^{\iota}xg_{j\alpha\beta\gamma}||_2 + ||\Lambda^{\iota}xg_{jk\alpha\beta\gamma}||_2 \lesssim (1+t)^{\delta}\varepsilon_0^3. \tag{3.7.52}$$

3.7.1 Terms generated by the vorticity

In the course of the above calculations, we generated some terms which are not present in [4] involving the vorticity. There are a large number of such terms (see in particular (3.7.20)), but they do not present any special difficulty and can be bounded in essentially the same way that we handled the term g_3 in Section 3.6.2. We record here some typical examples of the terms which are at most quadratic in the dispersive variable u, since terms with more nonlinear dependence on u are simpler to deal with.

A short calculation shows that the terms arising in g_3 are of the form:

$$\int_0^t \int_{\mathbb{R}^2} e^{is|\xi|^{1/2}} a_1(\xi) a_2(\xi-\eta) a_3(\eta) \hat{V}_\omega(s,\xi-\eta)\hat{u}(s,\eta)\, dsd\eta, \tag{3.7.53}$$

where a_1, a_2, a_3 are symbols satisfying the bounds:

$$\sup_{z\in\mathbb{R}^2} \big(|\partial^m a_1(z)| \min(|z|^{-1/2+m},1) + |\partial^m a_2(z)| \min(|z|^m,1) \\ + |\partial^m a_3(z)| \min(|z|^{-1/2+m},1)\big) \lesssim 1, \tag{3.7.54}$$

for $m = 0, 1$. Fixing $\alpha, \beta \in \{+,-\}$ and $1 \le j \le 2$, and writing $\mu = \mu_{j\alpha\beta}$, we see that we have terms of the following forms:

$$A^1_\omega = \int_0^t \int_{\mathbb{R}^2} e^{is\varphi(\xi,\eta)} \mu(\xi,\eta) \frac{\xi-\eta}{|\xi-\eta|} \cdot e^{\pm is|\xi-\eta|^{1/2}} \hat{V}_\omega(\xi-\eta)\hat{f}(s,\eta)\, d\eta \tag{3.7.55}$$

$$A^2_\omega = \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{is\varphi(\xi,\eta)} \mu(\xi,\eta) a_1(\xi-\eta) a_2(\xi-\eta-\sigma) a_3(\sigma) e^{-is|\xi-\eta-\sigma|^{1/2}} \\ \times \hat{V}_\omega(\xi-\eta-\sigma)\hat{f}(\sigma)\hat{f}(\eta)\, d\sigma d\eta ds, \tag{3.7.56}$$

and

$$A^3_\omega = \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{is\varphi(\xi,\eta)} \mu(\xi,\eta) |\xi-\eta|^{1/2} \frac{\xi-\eta-\sigma}{|\xi-\eta-\sigma|}\frac{\sigma}{|\sigma|} \\ \times \hat{V}_\omega(\xi-\eta-\sigma)\hat{V}_\omega(\sigma)\hat{f}(\eta)\, d\sigma d\eta ds, \tag{3.7.57}$$

as well as the following term, generated by (3.7.20) after integrating parts:

$$A^4_\omega = \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{is\varphi(\xi,\eta,\sigma)} \mu(\xi,\eta,\sigma) \frac{\xi-\eta-\sigma}{|\xi-\eta-\sigma|} \hat{V}_\omega(\xi-\eta-\sigma)\hat{f}(\sigma)\hat{f}(\eta)\, d\sigma d\eta ds, \tag{3.7.58}$$

with $\mu = \mu_{j\alpha\beta\gamma}$, the other terms being simpler to deal with.

Bounds for these terms can be proved in a nearly identical way to how we proved the bounds in Lemma 3.6.2, but using the estimates (3.7.3)-(3.7.4), (3.7.5)-(3.7.6) instead of Hölder's inequality.

3.8 Estimates for the higher-order terms coming from the Dirichlet-to-Neumann operator

It finally remains to bound the terms coming from $G_4(h)$:

$$G_4(h) \equiv G(h) - \Lambda - \nabla \cdot (h\nabla) - \Lambda(h\Lambda). \tag{3.8.1}$$

The terms coming from $G_4(h)$ can be handled by using the following lemma:

Lemma 3.8.1. *If the bootstrap assumption (3.2.9)-(3.2.11) hold with ε_0 sufficiently small, then:*

$$||\nabla^k G_4(h)g||_r \lesssim ||\nabla^{k+2}g||_p||\nabla^{k+2}u||_q||u||^2_{W^{4,\infty}}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}, 1<p,q<\infty, \tag{3.8.2}$$

and

$$||\Lambda^\iota x G_4(h)g||_2 \lesssim ||u||^2_{W^{4,\infty}}||\langle x\rangle^b u||_p||\langle x\rangle^a \Lambda g||_q, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}, 1<p,q<\infty, a+b=1, a,b\geq 0. \tag{3.8.3}$$

The estimate (3.8.2) follows from the results in [4] (see also [26]). The estimate (3.8.3) follows nearly directly from the arguments in [4], and we now sketch how to prove this estimate. As in Section 3.7, this section contains no ideas which are not essentially present in [4]. Let $\chi(r)$ be a smooth compactly supported function so that $\chi(r) = 0$ when $r \geq 2$ and $\chi(r) = 1$ when $r \leq 1$. Define:

$$K_n^1 g(x) = \nabla \int_{\mathbb{R}^2} \chi(|x-y|)\Lambda g(y)\frac{|h(x)-h(y)|^{2n}}{|x-y|^{2n+1}}\,dy \tag{3.8.4}$$

$$K_n^2 g(x) = \nabla \int_{\mathbb{R}^2} (1 - \chi(|x-y|))\Lambda g(y) \frac{|h(x) - h(y)|^{2n}}{|x-y|^{2n+1}}\, dy. \tag{3.8.5}$$

To avoid technical complications, we bound $||\Lambda^t F||_2 \lesssim ||F||_{H^1}$, and arguing as in [4], to prove (3.8.3), it will suffice to prove that for $j = 1, 2$ and $m \geq 2$:

$$||xK_n^j g||_{H^1} \lesssim ||u||^2_{W^{4,\infty}} ||u||_p ||\langle x \rangle g||_q. \tag{3.8.6}$$

To prove the estimate for $j = 1$, we expand out the product in the definition of K_n^1, and we need to control the L^2 norm of:

$$\begin{aligned}\int_{\mathbb{R}^2} x\Lambda g(y) \nabla_x^j \left(\chi(|x-y|) \frac{h(x)^\ell h(y)^{2n-\ell}}{|x-y|^{2n+1}}\right) dy &= \int_{\mathbb{R}^2} y\Lambda g(y) \nabla_x^j \left(\chi(|x-y|) \frac{h(x)^\ell h(y)^{2n-\ell}}{|x-y|^{2n+1}}\right) dy \\ &+ \int_{\mathbb{R}^2} \Lambda g(y) \nabla_x^j \left(\chi(|x-y|) \frac{h(x)^\ell h(y)^{2n-\ell}(x-y)}{|x-y|^{2n+1}}\right), \end{aligned} \tag{3.8.7}$$

for $j = 0, 1$ and $\ell = 0, ..., 2n$. These are simple to control because of the rapidly decaying kernel. When the derivatives fall on the factor of $\frac{1}{|x-y|^{2n+1}}$ the kernel decays faster and so the worst terms are:

$$\begin{aligned}&\int_{\mathbb{R}^2} y\Lambda g(y) \chi(|x-y|) \nabla^2 (h(x)^\ell) h(y)^{2n-\ell} \frac{1}{|x-y|^{2n+1}}\, dy \\ &\quad + \int_{\mathbb{R}^2} \Lambda g(y) \chi(|x-y|) \nabla^2 (h(x)^\ell) h(y)^{2n-\ell} \frac{x-y}{|x-y|^{2n+1}}\, dy, \\ &\equiv \nabla^2 (h(x))^\ell \Gamma * (x\Lambda g h^{2n-\ell}) + \nabla^2 (h(x))^\ell (x\Gamma) * (\Lambda g h^{2n-\ell}) \end{aligned} \tag{3.8.8}$$

By Young's inequality, putting the kernels $\Gamma, x\Gamma$ in L^1:

$$||\Gamma * (x\Lambda g h^{2n-\ell})||_2 + ||(x\Gamma) * (\Lambda g h^{2n-\ell})||_2 \lesssim ||h||^\ell_{W^{2,\infty}} ||x\Lambda g h^{2n-\ell}||_2 \tag{3.8.9}$$

from which the claim follows when $\ell < 2n$. For the case $\ell = 2n$, we still bound $||(x\Gamma) *$

$(\Lambda g)||_2 \lesssim ||\Lambda g||_2$, while to control the other term, we instead bound:

$$||h^{2n-1}\nabla^2 h\Gamma * (x\Lambda g)||_2 \lesssim ||h||_{W^{2,\infty}}^{2n-2}||h\Gamma * (x\Lambda g)||_2 \lesssim ||h||_{W^{2,\infty}}^{2n-2}||h||_p||\Gamma * (x\Lambda g)||_q, \quad (3.8.10)$$

with $1/p + 1/q = 1/2$, with a similar estimate for the term $h^{2n-2}(\nabla h)^2\Gamma * (x\Lambda g)$. This completes the bounds for K_m^1.

The bounds for K_m^2 are more involved, but they can be proved by following the argument in [4] and proceeding as above. The main technical estimate is the following, which can be proved by a simple modification of the proof of Lemma F.3 in [4]: If $A \geq 0, B \geq 1$ and $1/p + 1/q = 1/2, 1 < p, q \leq \infty$, then:

$$||x\nabla^A \Lambda f \nabla^B g||_2 \lesssim ||\Lambda^{1/2} f||_{W^{A+B+1,p}}||xg||_q + ||x\Lambda^{1/2} f||_p||g||_{W^{A+B+1,q}}, \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad (3.8.11)$$

$$||x\nabla^A \nabla f \nabla^B g||_2 \lesssim ||f||_{W^{A+B+2,p}}||\langle x\rangle g||_q + ||\langle x\rangle f||_p||g||_{W^{A+B+2,q}}. \quad (3.8.12)$$

Using this lemma, arguing as in [4] and using Young's inequality as above gives the result.

We now show how to control the terms:

$$A_\omega^5 = \int_0^t e^{is\Lambda^{1/2}} G_4(h) R \cdot V_\omega(s)\, ds, \quad (3.8.13)$$

and

$$A^5 = \int_0^t e^{is\Lambda^{1/2}} G_4(h) u(s)\, ds. \quad (3.8.14)$$

The other terms involving $G_4(h)$ are more nonlinear and thus simpler to bound. The L^∞ bounds we need for these terms are a straightforward consequence of (3.8.2).

$$||\Lambda^\iota x A_\omega^5||_2 \lesssim \int_0^t ||\Lambda^\iota x e^{is\Lambda^{1/2}} G_4(h) V_\omega||_2 \lesssim \int_0^t s||\Lambda^{-1/2+\iota} G_4(h) V_\omega||_2\, ds + \int_0^t ||\Lambda^\iota (x G_4(h) V_\omega)||_2\, ds. \quad (3.8.15)$$

This first term can be bounded using (3.8.2). To control the second term, we take p, q so that $1/p + 1/q = 1/2$ and $q = 2 + \delta_0$ for small δ_0. Then, using (3.A.8) to control $||u||_{W^{4,p}} \lesssim \varepsilon_0(1+s)^{-1/2}$, say, along with the bootstrap assumptions (3.2.9), we have that is bounded by:

$$\int_0^t \varepsilon_0^3 \frac{1}{(1+s)^{5/2}} ||xV_\omega(s)||_q \, ds \lesssim \varepsilon_0^3 \varepsilon_1 (1+t)^\delta, \tag{3.8.16}$$

by (3.4.3). Note that it was crucial that we take $q > 2$ for this estimate and this is why we needed the slightly more general (3.8.3) compared to the result in [4].

To control A^5, the argument is nearly identical but a little more technical. The main term we need to control is:

$$\int_0^t ||\Lambda^\iota x G_4(h) u(s)||_2 \, ds, \tag{3.8.17}$$

and for this we use (3.8.3) with $1/p + 1/q = 1/2$ where q is chosen so that $\iota = 1 - 2/q$ (so $q = 2 + O(\iota)$ and $p = O(1/\iota)$). This gives:

$$\begin{aligned} \int_0^t ||\Lambda^\iota x G_4(h) u(s)||_2 \, ds &\lesssim \int_0^t \frac{\varepsilon_0^3}{(1+s)^{5/2}} ||\langle x \rangle u||_q \\ &\lesssim \int_0^t \frac{\varepsilon_0^3}{(1+s)^{5/2}} \left(||u||_p + s||f||_{q_0} + ||\Lambda^\iota x f||_2 \right) ds \lesssim \varepsilon_0^4 (1+t)^\delta, \end{aligned} \tag{3.8.18}$$

where q_0 is taken so that $1/2 + \iota = 2/q_0 - 2/q$ (thus $q_0 = 4/3 - O(\iota)$), and where we have used $||f(s)||_{q_0} \lesssim s^{(2/q_0 - 2/q)\delta} ||xf(s)||_q = s^{(2/q_0 - 2/q)\delta} ||\Lambda^\iota x f(s)||_2$, which follows from Hölder's inequality.

3.9 Acknowledgments

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3.A Interpolation and Sobolev inequalities

In this section we will assume that $\partial\mathcal{D}_t$ is given by the graph of a function, $\partial\mathcal{D}_t = \{(x, h(t,x)), x \in \mathbb{R}^2\}$, and further that we have a bound for the second fundamental form and injectivity radius of $\partial\mathcal{D}_t$, as well as a bound for $|\nabla h|$:

$$|\theta| + \frac{1}{\iota_0} + |\nabla h| \leq K. \tag{3.A.1}$$

Note that $\theta \sim \nabla^2 h$. We then have the following Sobolev inequalities:

Lemma 3.A.1. *If* $||u||_{L^6(\mathcal{D}_t)} + ||Du||_{L^2(\mathcal{D}_t)} < \infty$, *then:*

$$||u||_{L^6(\mathcal{D}_t)} \leq C(K)||Du||_{L^2(\mathcal{D}_t)}. \tag{3.A.2}$$

If $u \in W^{k,p}(\mathcal{D}_t)$ *then for* $k > \frac{3}{p}$:

$$||u||_{L^\infty(\mathcal{D}_t)} \leq C(K)||u||_{W^{k,p}(\mathcal{D}_t)}, \tag{3.A.3}$$

and if $u \in W^{k,p}(\partial\mathcal{D}_t)$, *then for* $k > \frac{2}{p}$:

$$||u||_{L^\infty(\partial\mathcal{D}_t)} \leq C(K)||u||_{W^{k,p}(\partial\mathcal{D}_t)}, \tag{3.A.4}$$

These estimates all follow from the estimates in the appendix of [6]. The estimates there are all stated for the case of a bounded domain but it is clear that the proof goes through for an unbounded domain.

We will also need interpolation estimates on $\partial\mathcal{D}_t$ and $\mathcal{D}_t$:

Lemma 3.A.2. *Let* $2 \leq p \leq s \leq q \leq \infty$ *and* $0 \leq k \leq m$. *Suppose that:*

$$\frac{m}{s} = \frac{k}{p} + \frac{m-k}{q} \tag{3.A.5}$$

If α is a $(0, r)$ tensor then with $a = \frac{k}{m}$,

$$||\nabla^k \alpha||_{L^s(\partial \mathcal{D}_t)} \leq C ||\alpha||^{1-a}_{L^q(\partial \mathcal{D}_t)} ||\nabla^m \alpha||^a_{L^p(\partial \mathcal{D}_t)} \tag{3.A.6}$$

and if $|\theta| + \frac{1}{\iota_0} \leq K$, then:

$$\sum_{j=0}^{k} ||D^j \alpha||_{L^s(\mathcal{D}_t)} \leq C(K) ||\alpha||^{1-a}_{L^q(\mathcal{D}_t)} \bigg(\sum_{j=0}^{m} ||D^j \alpha||_{L^p(\mathcal{D}_t)} \bigg)^a. \tag{3.A.7}$$

Finally, we will use the following interpolation inequality which is Lemma 5.1 in [4]:

Lemma 3.A.3. *If $2 \leq p \leq \infty, k \leq N_0 + \frac{2}{p} - 1$, then:*

$$||\nabla^k u||_{L^p(\mathbb{R}^2)} \lesssim (1+t)^{-1+\frac{2}{p}+\sigma} \big((1+t) ||u(t)||_{W^{4,\infty}(\mathbb{R}^2)} + (1+t)^{-\delta} ||u(t)||_{H^{N_0}(\mathbb{R}^2)} \big), \tag{3.A.8}$$

where $\sigma = \sigma(k, p, N_0, \delta) = \frac{k}{N_0 + \frac{2}{p} - 1} (\delta - \frac{2}{p} + 1)$.

3.B Schauder and L^p estimates

The following result is well-known (see e.g. Theorem 7.3 in [27]):

Proposition 3.B.1. *If $f = F$ on $\partial \mathcal{D}_t$ and $\partial \mathcal{D}_t$ is given by the graph of $h : \mathbb{R}^2 \to \mathbb{R}$, then for $k \geq 2$:*

$$||f||_{C^{k,\alpha}(\mathcal{D}_t)} \leq C(||h||_{C^{k,\alpha}(\mathbb{R}^2)}) \big(||\Delta f||_{C^{k-2,\alpha}(\mathcal{D}_t)} + ||F||_{C^{k-2,\alpha}(\partial \mathcal{D}_t)} + ||f||_{L^\infty(\mathcal{D}_t)} \big). \tag{3.B.1}$$

We will also need standard L^p estimates (see e.g. Theorem 15.2 in [27]):

Proposition 3.B.2. *If $f \in W^{2,p}(\mathcal{D}_t)$, $f = F$ on $\partial \mathcal{D}_t$ and $\partial \mathcal{D}_t$ is given by the graph of $h : \mathbb{R}^2 \to \mathbb{R}$, , then:*

$$||f||_{W^{k,p}(\mathcal{D}_t)} \leq C(||h||_{C^k(\mathbb{R}^2)}) \big(||\Delta f||_{W^{k-2,p}(\mathcal{D}_t)} + ||F||_{W^{k-1/p}(\partial \mathcal{D}_t)} + ||f||_{L^p(\mathcal{D}_t)} \big). \tag{3.B.2}$$

3.C Estimates from harmonic analysis

We collect a few results that we will use frequently.

Lemma 3.C.1. • *If $1 < p \leq q < \infty$ and $\alpha = \frac{2}{p} - \frac{2}{q}$ then:*

$$||\Lambda^{-\alpha} f||_{L^q} \lesssim ||f||_{L^p} \tag{3.C.1}$$

• *If $1 < p < \infty$ and $s \geq 0$, then for any $1 < p_1, p_2, q_1, q_2 < \infty$ with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$,*

$$||\Lambda^s (fg)||_p \lesssim ||\Lambda^s f||_{p_1} ||g||_{p_2} + ||f||_{q_1} ||\Lambda^s g||_{q_2}. \tag{3.C.2}$$

The estimate (3.C.1) is known as the Hardy-Littlewood fractional integration lemma; for a proof, see [28]. For a proof of (3.C.2), see [29].

We will also use the following estimate for the Dirichlet-to-Neumann map, which is Proposition 2.2 from [4]. As mentioned there, this is not optimal (both in terms of the regularity assumed of h and the number of derivatives of φ on the right-hand side) but this will suffice for our purposes.

Proposition 3.C.1. *If $\varphi : \partial\Omega \to \mathbb{R}$ where $\partial\Omega$ is the graph of a function h with $h \in W^{4,\infty}(\mathbb{R}^2)$ then:*

$$||\mathcal{N} \varphi||_{W^{2,\infty}(\mathbb{R}^2)} \lesssim ||\nabla \varphi||_{W^{3,\infty}(\mathbb{R}^2)} + ||\Lambda^{1/2} \varphi||_{L^\infty(\mathbb{R}^2)}, \tag{3.C.3}$$

with implicit constant depending on $||h||_{W^{4,\infty}(\mathbb{R}^2)}$.

We will also need a few results related to the operator $e^{it\Lambda^{1/2}}$. We fix a function θ supported in $3/4 \leq |\xi| \leq 8/3$ so that:

$$\sum_{j \in \mathbb{Z}} \theta(\xi/2^j) = 1, \xi \neq 0. \tag{3.C.4}$$

We then define the projection operators:

$$P_j = \theta(D/2^j), \qquad P_{<j} = \sum_{k \leq j} P_k, \qquad P_{\geq j} = 1 - P_{<j}. \tag{3.C.5}$$

The homogeneous Besov norm $||\cdot||_{\dot{B}^s_{p,q}}$ is defined by:

$$||f||^q_{\dot{B}^s_{p,q}} = \sum_{j\in\mathbb{Z}} \left(2^{sj}||P_j f||_{L^p}\right)^q. \tag{3.C.6}$$

It is well-known that $||f||_{\dot{B}^s_{2,2}} = ||f||_{H^s}$. These spaces will occur here because $e^{it\Lambda^{1/2}}$ is bounded on the space $\dot{B}^{3/2}_{1,1}$:

Theorem 3.C.1 (Lemma A.1 of [4])**.** *The following dispersive estimate holds:*

$$||e^{it\Lambda^{1/2}} f||_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{t}||f||_{\dot{B}^{3/2}_{1,1}}. \tag{3.C.7}$$

This estimate follows from a scaling argument and a stationary phase estimate.

We also have the following estimate, which can be found in [4]:

Lemma 3.C.2. *If* $1 < p < 2$*, then*

$$||f||_{\dot{B}^{3/2}_{1,1}} \lesssim \sum_{\ell\le 2} ||\nabla^\ell (xf)||_2 + ||\nabla^\ell f||_2. \tag{3.C.8}$$

In particular:

$$||e^{it\Lambda^{1/2}} f||_\infty \lesssim \frac{1}{1+t}\left(\sum_{\ell\le 2} ||\nabla^\ell (xf)||_2 + ||\nabla^\ell f||_2\right) \tag{3.C.9}$$

3.D Elliptic systems

We follow the approach of [24] and [30]. First, we define the space Y to be closure of $C^\infty(\mathcal{D}_t)$ with respect to the norm:

$$||u||_Y \equiv ||u||_{L^6(\mathcal{D}_t)} + ||Du||_{L^2(\mathcal{D}_t)}. \tag{3.D.1}$$

We note that by the Sobolev inequality (3.A.2), Y is actually a Hilbert space with inner product:

$$(u,v)_Y \equiv \int_{\mathcal{D}_t} Du\cdot Dv. \tag{3.D.2}$$

The goal of this section is to construct a solution β to the system:

$$\operatorname{div}\beta = 0, \quad \text{in } \mathcal{D}_t, \tag{3.D.3}$$

$$\operatorname{curl}\beta = \alpha, \quad \text{in } \mathcal{D}_t, \tag{3.D.4}$$

$$\beta \cdot N = 0 \quad \text{on } \partial\mathcal{D}_t, \tag{3.D.5}$$

where $\alpha \in L^{6/5}(\mathcal{D}_t)$. Suppose for the moment that the following system has a unique weak solution β':

$$\Delta\beta' = \alpha \qquad \text{in } \mathcal{D}_t, \tag{3.D.6}$$

$$\gamma_j^i \beta_i' = 0 \qquad \text{on } \partial\mathcal{D}_t, \tag{3.D.7}$$

$$D_N\beta_N' = -H\beta_N', \quad \text{on } \partial\mathcal{D}_t, \tag{3.D.8}$$

where $D_N = N^j D_j$, $H = \operatorname{tr}\theta$ is the mean curvature of $\partial\mathcal{D}_t$ and $\beta_N' = N^i\beta_i'$. We now recall that by the definition of the second fundamental form we have $\operatorname{div}\beta'|_{\partial\mathcal{D}_t} = \operatorname{tr} D\beta'|_{\partial\mathcal{D}_t} = \operatorname{div}_{\partial\mathcal{D}_t}(\Pi\beta') + H\beta_N$. Taking the divergence of (3.D.6) and applying this formula shows that β' satisfies:

$$\Delta \operatorname{div}\beta' = 0 \quad \text{in } \mathcal{D}_t, \tag{3.D.9}$$

$$\operatorname{div}\beta' = 0 \quad \text{on } \partial\mathcal{D}_t, \tag{3.D.10}$$

so that $\operatorname{div}\beta' = 0$ in $\mathcal{D}_t$. In particular this implies that $\Delta\beta' = \operatorname{curl}^2\beta'$. If we then set $\beta = \operatorname{curl}\beta'$, it follows that β satisfies (3.D.3) and (3.D.4). To see that β satisfies (3.D.5), we just note that $N \cdot \operatorname{curl}\beta$ only inolves tangential derivatives of $\gamma \cdot \beta'$ and thus this vanishes if (3.D.7) holds. We also remark that this choice of β is actually unique; if β_1, β_2 satisfy (3.D.3)-(3.D.5) it follows that $\beta_1 - \beta_2 = D\phi$ for some harmonic function ϕ which satisfies a Neumann problem with zero boundary data and is thus a constant.

We now prove that (3.D.6)-(3.D.8) has a unique weak solution:

Proposition 3.D.1. *There is an $\epsilon^* \ll 1$ with the following property. If $\alpha \in L^{6/5}(\mathcal{D}_t)$ and the mean curvature $H = \operatorname{tr}\theta$ satisfies:*

$$||H||_{L^3(\partial\mathcal{D}_t)} \leq \epsilon^*, \tag{3.D.11}$$

then the problem (3.D.6)-(3.D.8) *has a unique solution $\beta' \in H^1(\mathcal{D}_t)$. Furthermore, with $\beta = \operatorname{curl}\beta'$, under the above hypotheses we have:*

$$|\beta(z)| \lesssim \frac{1}{(1+|z|)^2}\int_{\mathcal{D}_t}(1+|z'|)|\alpha(z')|\,dz' \tag{3.D.12}$$

Proof. We let $C^\infty_{\tan}(\mathcal{D}_t)$ denote the collection of smooth one-forms α on $\mathcal{D}_t$ so that $\gamma_i^j\alpha_j$ is compactly supported in $\mathcal{D}_t$, and we let Y_0 denote the closure of $C^\infty_{\tan}(\mathcal{D}_t)$ with respect to the norm $||u||_Y = ||u||_{L^6(\mathcal{D}_t)} + ||Du||_{L^2(\mathcal{D}_t)}$. We define the bilinear form:

$$B[u,\varphi] \equiv \int_{\mathcal{D}_t}\delta^{ik}\delta^{j\ell}(D_iu_j)(D_k\varphi_\ell) + \int_{\partial\mathcal{D}_t}Hu_N\varphi_N, \tag{3.D.13}$$

for $u, \varphi \in Y_0$ and with $w_N = N^iw_i$. We want to find $u \in Y_0$ so that:

$$B[u,\varphi] = \int_{\mathcal{D}_t}\delta^{ij}\alpha_i\varphi_j, \tag{3.D.14}$$

for all $\varphi \in Y_0$. The map $\varphi \mapsto \int_{\mathcal{D}_t}\alpha\cdot\varphi$ is a continuous linear map on Y since $\alpha \in L^{6/5}(\mathcal{D}_t)$, and so by the Lax-Milgram theorem it suffices to prove that B is bounded and coercive.

Let d denote the geodesic distance to $\partial\mathcal{D}_t$ in Lagrangian coordinates and let $n = \partial d$ be the unit normal to $\partial\mathcal{D}_t$. Let χ denote a cutoff function which is 1 when $d < \rho$ and 0 when $d > 2\rho$, for fixed $\rho > 0$, and set $\tilde{n} = \chi n$. By Stokes' theorem we have:

$$\int_{\partial \mathcal{D}_t} H|w|^2 \, dS = \int_{\partial \mathcal{D}_t} \tilde{n}^k (\tilde{n}_k (H|w|^2) \, dS$$

$$= \int_{\mathcal{D}_t} \nabla_k (\chi \tilde{n}^k) H|w|^2 \, dxdy + \int_{\mathcal{D}_t} \chi H (\nabla_{\tilde{n}} w) \cdot w \, dxdy. \quad (3.D.15)$$

Since d is the geodesic distance, $\nabla_n \chi(d) = 0$ and this also implies that $\operatorname{div} \tilde{n} = -\chi H$. Therefore, from the above we have:

$$\left| \int_{\partial \mathcal{D}_t} H|w|^2 \, dS \right| \lesssim ||H^2||_{L^{3/2}(\mathbb{R}^2)} || \, |w|^2 \, ||_{L^3(\mathcal{D}_t)} + ||Hw||_{L^2(\mathcal{D}_t)} ||\nabla w||_{L^2(\mathcal{D}_t)}$$

$$\lesssim ||H||^2_{L^3(\mathbb{R}^2)} ||w||^2_{L^6(\mathcal{D}_t)} + ||H||_{L^3(\mathbb{R}^2)} ||w||_{L^6(\mathcal{D}_t)} ||\nabla w||_{L^2(\mathcal{D}_t)}. \quad (3.D.16)$$

In particular this shows that the bilinear form B is bounded on Y and also, provided $||H||_{L^3}$ is taken sufficiently small, that it is coercive on Y.

We now prove the decay estimate (3.D.12). For this, we will construct a Green's function G for the problem (3.D.6)-(3.D.8), following the approach of [25] and [30]. We fix $\rho > 0$ and let $G_\rho = G_\rho(z, z')$ denote the weak solution to the problem (3.D.6)-(3.D.8) with $\alpha_i = \frac{1}{|\rho|^3} \chi_{B_\rho(z')}(z)$, $i = 1, 2, 3$, where $B_\rho(z')$ denotes the ball of radius ρ centered at z' and χ is the cutoff function supported on this ball. Following the argument in section 4 of [25], one can prove that

$$||G_\rho(z, \cdot)||_{W^{1,p}(B_{d_z(z)})} \leq C(d_y), \quad (3.D.17)$$

for some p with $p \in (1, 3/2)$ and where d_z denotes the distance from z to $\partial \mathcal{D}_t$. The constant here depends on $||h||_{C^1(\mathbb{R}^2)}$. Taking a diagonal subsequence, for each z we get a function $G(z, \cdot) \in W^{1,p}(B_{d_z}(z))$ with $G_\rho(z, \cdot) \to G(z, \cdot)$ weakly in $W^{1,p}(B_{d_z}(z))$. We would like to conclude the following two estimates:

$$|G(z,z')| \leq C|z-z'|^{-1}, \quad |D_z G(z,z')| \leq C|z-z'|^{-2}, \tag{3.D.18}$$

where $C = C(||h||_{W^{1,\infty}(\mathbb{R}^2)})$. These estimates follows as in Section 5 of [30] and Theorem 3.13 in [31], provided that the system (3.D.6)-(3.D.8) satisfies the condition "(LH)" in [31]. However this follows from Corollary 4.9 there provided that the system (3.D.6)-(3.D.8) is sufficiently close to a diagonal system. Since we are assuming that $||h||_{W^{4,\infty}(\mathbb{R}^2)}$ is small, this follows after straightening the boundary. We can now prove (3.D.12). We can assume $|z| \geq 1$. If $|z'| \leq \frac{1}{2}|z|$, then $|z-z'| \geq \frac{1}{2}|z|$ so that:

$$\Big| \int_{\mathcal{D}_t} D_z G(z,z')\alpha(z')\, dz' \Big| \lesssim \int_{\mathcal{D}_t} \frac{1}{|z-z'|^2}|\alpha(z')|\, dz' \frac{1}{|z|^2}||\alpha||_{L^1(\mathcal{D}_t)}. \tag{3.D.19}$$

When $|z'| \geq \frac{1}{2}|z|$, we instead estimate::

$$\Big| \int_{\mathcal{D}_t} D_z G(z,z')\alpha(z')\, dz' \Big| \lesssim \int_{\mathcal{D}_t} \frac{1}{|z'|^2}|\alpha(z-z')|\, dz' \lesssim \int_{\mathcal{D}_t} \frac{1}{|z|^2}|\alpha(z-z')|\, dz'$$

$$\lesssim \frac{1}{|z|^2}||\alpha||_{L^1(\mathcal{D}_t)}, \tag{3.D.20}$$

as required. □

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EDUCATION

Johns Hopkins University Baltimore, Maryland
PhD. Candidate, Mathematics (expected May 2019)

- Research Topic: The free boundary problem for Euler's equations with vorticity
- Advisor: Dr. Hans Lindblad

University of Toronto, Toronto, Ontario

- MSc., Mathematics, May 2013

  Research Topic: Stability of the Abrikosov Lattice Solutions to the Ginzburg-Landau equations

  Advisor: Dr. I.M. Sigal

- H. B. Sc. Specialist, Mathematics; Major, Physics, 2007-2012

RESEARCH INTERESTS

Partial differential equations, Water waves, General relativity

PUBLICATIONS

D. Ginsberg, H. Lindblad, and C. Luo, *Local well-posedness for the motion of a compressible, self-gravitating liquid with free surface boundary*, Preprint `arXiv:1902.08600`

D. Ginsberg, *On the breakdown of solutions to the incompressible Euler equations with free surface boundary*, Preprint `arXiv:1811.06154`

D Ginsberg, *A priori estimates for the relativistic Euler equations with free surface boundary*, Preprint `arxiv.org:1811.06915`

D Ginsberg, *On the lifespan of three-dimensional gravity water waves with vorticity*, Preprint `arxiv.org:1812.01583`

D. Ginsberg and G. Simpson, *Analytical and Numerical Results on the Positivity of Steady State Solutions of a Thin Film Equation*, DCDS-B, 18(5):1305-1321, 2013. (Undergraduate work)

INVITED TALKS

A priori estimates for a relativistic liquid, SIAM Conference on Analysis of Partial Differential Equations, Baltimore, MD, December 2017.

SELECTED AWARDS

- *William Kelso Morrill Award*, May 2018. Teaching award.
- *Professor Joel Dean Award*, May 2016. Teaching award.
- *Krieger School of Arts and Sciences, "Owens Fellowship"*, Sept. 2013-Sept. 2016. Awarded to the department's most competitive applicants.
- *NSERC Postgraduate Scholarship*, Sept. 2013-May 2014.

Teaching Experience

Johns Hopkins University: Instructor
Math 302: Differential equations with applications, Summer 2018 (online)
Math 302: Differential equations with applications, Summer 2016 (online)
Math 302: Differential equations with applications, Summer 2015 (online)
Math 108: Calculus 1 (Physical Sciences & Engineering), Summer 2014.

Johns Hopkins University: Graduate Teaching Assistant
Math 439: Introduction to differential geometry, Fall 2018.
Math 201: Linear algebra, Spring 2018
Math 109: Calculus for physical sciences and engineering, Fall 2017
Math 302: Differential equations with applications, Spring 2017
Math 201: Linear algebra, Fall 2016
Math 302: Differential equations with applications, Spring 2016
Math 302: Differential equations with applications, Fall 2015
Math 109: Calculus for physical sciences and engineering, Spring 2015
Math 108: Calculus 1, Fall 2014
Math 108: Calculus 1, Fall 2013

University of Toronto, Teaching Assistant
Math 337: Introduction to Real Analysis, Fall 2012
Math 133: Calculus and linear algebra for commerce students, Fall 2012 and Spring 2013
Math 188: Linear algebra for engineering students, Fall 2012
Math 135: Calculus for life sciences, Fall 2010-Spring 2012